

Homework 3 Solutions

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Other Problems

Problem 4

We begin by proving a quick lemma.

Lemma 1. *A function $f : A \rightarrow B$ is a bijection if and only if there is a function $g : B \rightarrow A$ such that $g(f(a)) = a$ for all $a \in A$ and $f(g(b)) = b$ for all $b \in B$. The function g is called the inverse of f .*

Proof. Suppose $f : A \rightarrow B$ is a bijection. Then let $g : B \rightarrow A$ be the function defined as follows: for any $b \in B$, there exists $a \in A$ such that $f(a) = b$ by surjectivity of f . We let $g(b) = a$. This is well-defined since if $f(a_1) = f(a_2) = b$, then $a_1 = a_2$ by injectivity, so $g(b) = a_1 = a_2$ is a unique element. Then by construction $g(f(a)) = a$. Moreover $f(g(b)) = f(a)$ where a is an element such that $f(a) = b$, so $f(g(b)) = b$.

Now suppose $f : A \rightarrow B$ has an inverse g . Then if $f(a_1) = f(a_2)$, we apply g to see that $a_1 = g(f(a_1)) = g(f(a_2)) = a_2$, so f is injective. Moreover, if $b \in B$, we have $f(g(b)) = b$, so b is $f(a)$ for $a = g(b)$. Ergo f is surjective. So f is a bijection. \square

(a) Consider the map $f : [0, 1] \rightarrow [0, 1]$ defined by

$$f(x) = \begin{cases} f(\frac{1}{2^{n-1}}) = \frac{1}{2^n} \\ f(x) = x & x \neq \frac{1}{2^{n-1}} \end{cases}$$

This has inverse $g : [0, 1] \rightarrow [0, 1]$ given by

$$f(x) = \begin{cases} f(\frac{1}{2^n}) = \frac{1}{2^{n-1}} \\ f(x) = x & x \neq \frac{1}{2^n} \end{cases}$$

hence is a bijection.

(b) From class, $f(x) = \frac{x}{x^2+1}$ is a bijection from $(-1, 1)$ to \mathbb{R} . We also have the map $g : (a, b) \rightarrow (-1, 1)$ which is defined by $g(x) = \frac{2(x-a)}{b-a} - 1$, which has inverse $h : (-1, 1) \rightarrow (a, b)$ given by $h(x) = \frac{(b-a)(x+1)}{2}$. Then $f \circ g$, the function given by mapping x to $f(g(x))$, is the desired bijection.

(c) Let S be the set of sequences (a_n) such that $a_n \in \{0, 1\}$ and $P(\mathbb{N})$ be the set of subsets of \mathbb{N} . Consider the map $f : S \rightarrow P(\mathbb{N})$ which sends (a_n) to the subset $A \subset \mathbb{N}$ such that $n \in A$ if and only if $a_n = 1$. This has a clear inverse by reversing the operation, hence is a bijection.

Section 1.5

Problem 1.5.1

Assume B is countable. Then there is a bijection $f : \mathbb{N} \rightarrow B$. Let A be an infinite subset of B . Then set $n_1 = \min\{n \in \mathbb{N} : f(n) \in A\}$. Continue inductively, defining n_i to be $\min\{n \in \mathbb{N} : f(n) \in A, n_i > n_{i-1}\}$. Then $g : \mathbb{N} \rightarrow A$ defined by $g(i) = f(n_i)$ is a map from \mathbb{N} to A which is injective because f is injective and each of the n_i are distinct, and onto because f is onto, hence a bijection.

Problem 1.5.9

(a) We see that $\sqrt{2}$ is a root of $x^2 - 2 = 0$ and that $\sqrt[3]{2}$ is a root of $x^3 - 2 = 0$. If $x = \sqrt{2} + \sqrt{3}$, we notice that $x^2 = 5 + 2\sqrt{6}$, so $(x^2 - 5)^2 = 24$. Expanding this out we see that $x^4 - 10x^2 - 49 = 0$. So $\sqrt{2} + \sqrt{3}$ is a root of this polynomial.

(b) For $n \in \mathbb{N}$, let P_n be the set of polynomials with integer coefficients having degree n . Notice that P_n is equivalent to an ordered tuple of integers (a_n, \dots, a_0) , or an element of the product \mathbb{Z}^n . Since \mathbb{Z} is countable, we may apply the usual counting-along-diagonals argument n times to show that the set of all such tuples of integers is countable, so the set P_n is countable. Now we let A_n be the set of all roots of polynomials in P_n . The total number of roots of a polynomial of degree n is finite, so A_n is a union of a countable number (one for each element of P_n) of finite sets. By Theorem 1.5.8, A_n itself is countable.

(c) The set of algebraic numbers is $\bigcup_{n=1}^{\infty} A_n$, which is a countable union of countable sets, hence countable. We conclude that the set of transcendental numbers must be uncountable.

Section 2.2

Problem 2.2.2

(a) Let $\epsilon > 0$, and choose N such that $N < \frac{25\epsilon}{6}$. Then for $n \geq N$, we have

$$\left| \frac{2n+1}{5n+4} - \frac{2}{5} \right| = \left| \frac{10n+2 - (10n+8)}{5(5n+4)} \right| = \frac{6}{25n+20} < \frac{6}{25n} < \epsilon.$$

So $\lim_{n \rightarrow \infty} \frac{2n+1}{5n+4} = \frac{2}{5}$.

(b) Let $\epsilon > 0$. Choose N such that $\frac{\epsilon}{2} < N$. Then for $n \geq N$, we have

$$\left| \frac{2n^2}{n^3+3} - 0 \right| = \left| \frac{2n^2}{n^3+3} \right| < \left| \frac{2n^2}{n^3} \right| = 2 \cdot \frac{1}{n} < 2 \cdot \frac{\epsilon}{2} = \epsilon.$$

So $\lim_{n \rightarrow \infty} \frac{2n^2}{n^3+3} = 0$.

(c) Recall that $|\sin x| \leq 1$ for all real x . Given $\epsilon > 0$, let N be a natural number such that $N > \frac{1}{\epsilon^3}$. Then for $n \geq N$, we have

$$\left| \frac{\sin(n^2)}{\sqrt[3]{n}} - 0 \right| = \frac{|\sin(n^2)|}{\sqrt[3]{n}} \leq \frac{1}{\sqrt[3]{n}} \leq \frac{1}{\sqrt[3]{N}} < \epsilon.$$

So $\lim_{n \rightarrow \infty} \frac{\sin(n^2)}{\sqrt[3]{n}} = 0$.

Problem 2.2.3

(a) We would have to find an example of a college in the United States at which every student is less than seven feet tall.

(b) We would have to find a college in the United States at which every professor gives a grade lower than a B to some students (or has given a grade of lower than a B to at least one student in the past, the phrasing is mildly ambiguous).

(c) We would have to show that every college in the United States has a student who is less than six feet tall.

Section 2.3

Problem 2.3.2

Let $x_n \rightarrow 2$.

(a) Given $\epsilon > 0$, choose N such that $n \geq N$ implies that $|x_n - 2| < \frac{3\epsilon}{2}$. Then for $n \geq N$,

$$\left| \frac{2x_n - 1}{3} - 1 \right| = \left| \frac{2x_n - 4}{3} \right| = \frac{2|x_n - 2|}{3} < \frac{2}{3} \cdot \frac{3\epsilon}{2} = \epsilon.$$

So $\lim_{n \rightarrow \infty} \frac{2x_n - 1}{3} = 1$.

(b) First observe that

$$\begin{aligned} \left| \frac{1}{x_n} - \frac{1}{2} \right| &= \left| \frac{2 - x_n}{2x_n} \right| \\ &= \frac{|2 - x_n|}{2|x_n|} \end{aligned}$$

Now choose N_1 such that $n \geq N_1$ implies that $|x_n - 2| < 1$, which is to say $-1 < x_n - 2 < 1$, so that $1 < x_n < 3$, and in particular $x_n > 1$. Then given $\epsilon > 0$, choose N_2 such that $n \geq N_2$ implies that $|2 - x_n| < 2\epsilon$. Then $n \geq N = \max\{N_1, N_2\}$ implies that

$$\begin{aligned} \left| \frac{1}{x_n} - \frac{1}{2} \right| &= \left| \frac{2 - x_n}{2x_n} \right| \\ &= \frac{|2 - x_n|}{2|x_n|} \\ &< \frac{2\epsilon}{2(1)} \\ &= \epsilon. \end{aligned}$$

We see that $\frac{1}{x_n} \rightarrow \frac{1}{2}$ as desired.

Problem 2.3.4

(a) Since $a_n \rightarrow 0$, the first three algebraic limit theorems imply that $\lim(1+2a_n) = 1+2(0) = 0$ and $\lim(1+3a_n-4a_n^2) = 1+3(0)-4(0)^2 = 1$. Since in particular the second number is nonzero, we now have that

$$\lim \left(\frac{1+2a_n}{1+3a_n-4a_n^2} \right) = \frac{1+0}{1+0} = 1.$$

(b) We observe that

$$\frac{(a_n+2)^2-4}{a_n} = \frac{a_n^2+4a_n+4-4}{a_n} = \frac{a_n^2+4a_n}{a_n} = a_n+4.$$

The algebraic limit theorems imply that $\lim(a_n+4) = 0+4 = 4$, so

$$\lim \left(\frac{(a_n+2)^2-4}{a_n} \right) = 4.$$

(c) We observe that

$$\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5} = \frac{2+3a_n}{1+5a_n}.$$

The algebraic limit theorems imply that the limit of the righthand term is 2 as $a_n \rightarrow 0$. Ergo

$$\lim \left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5} \right) = 2.$$

Problem 2.3.8

(a) Let $p(x) = c_mx^m + \dots + c_1x + c_0$ be a polynomial and let $x_n \rightarrow x$. Then the algebraic limit theorems imply that $p(x_n) = c_mx_n^m + \dots + c_1x_n + c_0 \rightarrow c_mx^m + \dots + c_1x + c_0 = p(x)$.

(b) Consider the function $f : [0, 1] \rightarrow \{0, 1\}$ defined by

$$f(x) = \begin{cases} 1 & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Consider the sequence $x_n = \frac{1}{n}$, which converges to $x = 0$. Then $f(x_n) = 1$ but $f(x) = 0$, so $f(x_n)$ does not converge to $f(x)$.