Section 1.2

Problem 1.2.3

(a) This is false; let \( A_n = \{ n, n+1, n+2, \ldots \} \) for \( n \in \mathbb{N} \) so that \( A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots \). Then \( \bigcap_{n=1}^\infty A_n = \emptyset \) and in particular is not infinite.

(b) This is true (finiteness is important).

(c) This is false. Consider \( A = B = \{1\} \) and \( C = \{2\} \). Then \( A \cap (B \cup C) = \{1\} \) but \( (A \cap B) \cup C = \{1,2\} \).

(d) This is true.

(e) This is true.

1.2.5(c)

We want to show that if \( A, B \subseteq C \), then \( (A \cup B)^c = A^c \cap B^c \).

First let \( x \in (A \cup B)^c \). Then \( x \notin A \cup B \), which means that \( x \notin A \) and \( x \notin B \). Hence \( x \in A^c \) and \( x \in B^c \), so \( x \in A^c \cap B^c \). As \( x \) was arbitrary, we have that \( (A \cup B)^c \subseteq A^c \cap B^c \).

In the other direction, let \( x \in A^c \cap B^c \). Then \( x \in A^c \) and \( x \in B^c \), implying that \( x \notin A \) and \( x \notin B \). Hence \( x \notin A \cup B \), and therefore \( x \in (A \cup B)^c \). As \( x \) was arbitrary, we have that \( A^c \cap B^c \subseteq (A \cup B)^c \).

As we have shown inclusions in both directions, we conclude that the two sets are equal.

1.2.11

(a) There exists a pair of real numbers \( a < b \) such that \( a + \frac{1}{n} \geq b \) for all \( n \in \mathbb{N} \). The claim is the true statement, this negation is false.

(b) For any real number \( x > 0 \), there is some \( n \in \mathbb{N} \) such that \( \frac{1}{n} \leq x \). This is true, the original was false.

(c) There exists a pair of real numbers \( a < b \) such that every real number \( c \) such that \( a < c < b \) is irrational. The claim is the true statement, this negation is false.
Section 1.3

1.3.5

Let $A$ be nonempty and bounded above, and let $c \in \mathbb{R}$. Set $cA = \{ca : a \in A\}$.

(a) We claim that if $c \geq 0$, then $\sup cA = c \sup A$. First, the statement is obviously true if $c = 0$ since then $cA = \{0\}$ and $c \sup A = 0$. Now let $c > 0$. Let $s = \sup A$. Then $a \leq s$ for all $a \in A$, so $ca \leq cs$ for all $ca \in cA$, so $cs$ is an upper bound for $cA$. Suppose $u$ is an arbitrary upper bound for $cA$. Then $ca \leq u$ for all $ca \in cA$, implying that $a \leq \frac{u}{c}$ for all $a \in A$. Ergo $\frac{u}{c}$ is an upper bound for $A$ and in particular $s \leq \frac{u}{c}$. So $cs \leq u$. Hence $cs$ is an upper bound for $cA$ which is less than or equal to any upper bound for $cA$, and therefore $cs = \sup(cA)$. So $c \sup A = \sup cA$.

(b) We conjecture that if $A$ is nonempty and bounded below and $c < 0$, $c \sup A = \inf cA$. See the final problem for the case that $c = -1$.

1.3.6

Given $A$ and $B$ subsets of the real line, we set $A + B = \{a + b : a \in A, b \in B\}$. Let $A$ and $B$ be bounded above with $s = \sup A$ and $t = \sup B$.

(a) First we observe that since $a \leq s$ for any $a \in A$ and $b \leq t$ for any $b \in B$, for any $a + b$ in $A + B$ we have $a + b \leq s + t$, so $s + t$ is an upper bound for $A + B$.

(b) Now let $u$ be any upper bound for $A + B$. Fix $a \in A$. We see that $a + b \leq u$ for all $b \in B$, implying that $b \leq u - a$ for all $b \in B$. Hence $u - a$ is an upper bound for $B$, and in particular $t \leq u - a$ since $t$ is less than or equal to any upper bound for $B$.

(c) Rearranging $t \leq u - a$ for all $a \in A$, we see that $a \leq u - t$ for all $a \in A$. So $u - t$ is an upper bound for $A$ and in particular $s \leq u - t$. Ergo $s + t \leq u$. Since $s + t$ is an upper bound for $A + B$ which is less than or equal to any upper bound for $A + B$, it must be the supremum of $A + B$. So $\sup(A + B) = \sup A + \sup B$.

(d) Alternately, we can use the characterization of the supremum given in Lemma 1.3.8, starting from the fact that $s + t$ is an upper bound for $A + B$ which we showed in part (a). Let $\epsilon > 0$. Since $s = \sup A$, there is some $a \in A$ such that $a > s - \frac{\epsilon}{2}$. Similarly since $t = \sup B$, there is some $b \in B$ such that $b > t - \frac{\epsilon}{2}$. Then we observe that $a + b > s - \frac{\epsilon}{2} + b > t - \frac{\epsilon}{2} = (s + t) - \epsilon$. Since $\epsilon$ was arbitrary, we conclude that for all $\epsilon > 0$ there is an element of $A + B$ which is greater than $(s + t) - \epsilon$, and therefore that $s + t = \sup(A + B)$.

1.3.8

The suprema and infima of the sets are as follows.

(a) $\left\{\frac{m}{n} : m, n \in \mathbb{N}, m < n\right\}$. The infimum is zero and the supremum is one, by noticing that $0 < \frac{m}{n} < 1$ for all elements of the set and that $\frac{1}{n}$ can be made arbitrarily close to 0 and $\frac{n-1}{n}$ can be made arbitrarily close to 1.
(b) \( \left\{ \frac{(-1)^m}{n} : m, n \in \mathbb{N} \right\} \). The infimum, which is also a minimum, is \(-1\) and the supremum, which is also a maximum, is \(1\).

(c) \( \left\{ \frac{n}{3n+1} : n \in \mathbb{N} \right\} \). This is an increasing sequence; the infimum, which is also a minimum, is \(\frac{1}{4}\) and the supremum is \(\frac{1}{3}\).

(d) \( \left\{ \frac{m}{m+n} : m, n \in \mathbb{N} \right\} \). This is the same set as part (a)!

Section 1.4

1.4.5

We claim that if \(a\) and \(b\) are real numbers with \(a < b\), there is an irrational number \(t\) such that \(a < t < b\). Consider the real numbers \(a - \sqrt{2} < b - \sqrt{2}\). By the density of the rationals in the reals, there is a rational \(r\) such that \(a - \sqrt{2} < r < b - \sqrt{2}\). Adding \(\sqrt{2}\) to all three terms we see that \(a < r + \sqrt{2} < b\). Now by Exercise 1.4.1 from workshop, the sum of a rational number and an irrational number is always irrational, so in particular \(r + \sqrt{2}\) is irrational. Therefore we are done.

Other Problems

Problem 6

Let \(S\) be nonempty and bounded below, say by \(\ell\). Then for any \(s \in S\), we have \(\ell \leq s\), implying that \(-s \leq \ell\). So \(\ell\) is an upper bound for \(-S\). Since \(-S\) is bounded above it has a supremum by the Axiom of Completeness, call it \(a\). We claim that \(-a\) is the infimum of \(S\). First, since \(a\) is an upper bound of \(-S\), we have \(-s \leq a\) for all \(-s \in -S\), so we see that \(-a \leq s\) for all \(s \in S\), hence \(-a\) is a lower bound for \(S\). Furthermore suppose that \(\ell\) is an arbitrary lower bound for \(S\), then as we argued above \(-\ell\) is an upper bound for \(-S\), so \(a \leq -\ell\), implying that \(\ell \leq -a\). Hence \(a\) is greater than or equal to any lower bound for \(S\). Ergo \(a\) is the infimum of \(S\). We conclude that bounded below nonempty subsets of the real numbers have infima.