Homework 2 Solutions

February 3, 2021

Section 1.2

Problem 1.2.3

(a) This is false; let $A_n = \{n, n+1, n+2, \dots\}$ for $n \in \mathbb{N}$ so that $A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots$. Then $\bigcap_{n=1}^{\infty} A_n = \emptyset$ and in particular is not infinite.

(b) This is true (finiteness is important).

(c) This is false. Consider $A = B = \{1\}$ and $C = \{2\}$. Then $A \cap (B \cup C) = \{1\}$ but $(A \cap B) \cup C = \{1, 2\}$.

(d) This is true.

(e) This is true.

1.2.5(c)

We want to show that if $A, B \subset C$, then $(A \cup B)^c = A^c \cap B^c$.

First let $x \in (A \cup B)^c$. Then $x \notin A \cup B$, which means that $x \notin A$ and $x \notin B$. Hence $x \in A^c$ and $x \in B^c$, so $x \in A^c \cap B^c$. As x was arbitrary, we have that $(A \cup B)^c \subseteq A^c \cap B^c$.

In the other direction, let $x \in A^c \cap B^c$. Then $x \in A^c$ and $x \in B^c$, implying that $x \notin A$ and $x \notin B$. Hence $x \notin A \cup B$, and therefore $x \in (A \cup B)^c$. As x was arbitary, we have that $A^c \cap B^c \subseteq (A \cup B)^c$.

As we have shown inclusions in both directions, we conclude that the two sets are equal.

1.2.11

(a) There exists a pair of real numbers $a < b$ such that $a + \frac{1}{n} \geq b$ for all $n \in \mathbb{N}$. The claim is the true statement, this negation is false.

(b) For any real number $x > 0$, there is some $n \in \mathbb{N}$ such that $\frac{1}{n} \leq x$. This is true, the original was false.

(c) There exists a pair of real numbers $a < b$ such that every real number c such that $a < c < b$ is irrational. The claim is the true statement, this negation is false.

Section 1.3

1.3.5

Let A be nonempty and bounded above, and let $c \in \mathbb{R}$. Set $cA = \{ca : a \in A\}$.

(a) We claim that if $c \geq 0$, then $\sup cA = c \sup A$. First, the statement is obviously true if $c = 0$ since then $cA = \{0\}$ and $c \sup A = 0$. Now let $c > 0$. Let $s = \sup A$. Then $a \leq s$ for all $a \in A$, so $ca \leq cs$ for all $ca \in cA$, so cs is an upper bound for cA. Suppose u is an arbitrary upper bound for cA. Then $ca \leq u$ for all $ca \in cA$, implying that $a \leq \frac{u}{c}$ for all $a \in A$. Ergo $\frac{u}{c}$ is an upper bound for A and in particular $s \leq \frac{u}{c}$. So $cs \leq u$. Hence cs is an upper bound for cA which is less than or equal to any upper bound for cA, and therefore $cs = \sup(cA)$. So $c \sup A = \sup cA$.

(b) We conjecture that if A is nonempty and bounded below and $c < 0$, $c \sup A = \inf cA$. See the final problem for the case that $c = -1$.

1.3.6

Given A and B subsets of the real line, we set $A + B = \{a + b : a \in A, b \in B\}$. Let A and B be bounded above with $s = \sup A$ and $t = \sup B$.

(a) First we observe that since $a \leq s$ for any $a \in A$ and $b \leq t$ for any $b \in B$, for any $a + b$ in $A + B$ we have $a + b \leq s + t$, so $s + t$ is an upper bound for $A + B$.

(b) Now let u be any upper bound for $A + B$. Fix $a \in A$. We see that $a + b \leq u$ for all $b \in B$, implying that $b \leq u - a$ for all $b \in B$. Hence $u - a$ is an upper bound for B, and in particular $t \leq u - a$ since t is less than or equal to any upper bound for B.

(c) Rearranging $t \leq u - a$ for all $a \in A$, we see that $a \leq u - t$ for all $a \in A$. So $u - t$ is an upper bound for A and in particular $s \leq u - t$. Ergo $s + t \leq u$. Since $s + t$ is an upper bound for $A + B$ which is less than or equal to any upper bound for $A + B$, it must be the supremum of $A + B$. So $\sup(A + B) = \sup A + \sup B$.

(d) Alternately, we can use the characterization of the supremum given in Lemma 1.3.8, starting from the fact that $s + t$ is an upper bound for $A + B$ which we showed in part (a). Let $\epsilon > 0$. Since $s = \sup A$, there is some $a \in A$ such that $a > s - \frac{\epsilon}{2}$. Similarly since $t = \sup B$, there is some $b \in B$ such that $b > t - \frac{\epsilon}{2}$. Then we observe that $a + b > s - \frac{\epsilon}{2} + b > t - \frac{\epsilon}{2} = (s + t) - \epsilon$. Since ϵ was arbitrary, we conclude that for all $\epsilon > 0$ there is an element of $A + B$ which is greater than $(s + t) - \epsilon$, and therefore that $s + t = \sup(A + B)$.

1.3.8

The suprema and infima of the sets are as follows.

(a) $\left\{\frac{m}{n} : m, n \in \mathbb{N}, m < n\right\}$. The infimum is zero and the supremum is one, by noticing that $0 < \frac{m}{n} < 1$ for all elements of the set and that $\frac{1}{n}$ can be made arbitrarily close to 0 and $\frac{n-1}{n}$ can be made arbitrarily close to 1.

(b) $\left\{\frac{(-1)^m}{n} : m, n \in \mathbb{N}\right\}$. The infimum, which is also a minimum, is -1 and the supremum, which is also a maximum, is 1.

(c) $\left\{\frac{n}{3n+1} : n \in \mathbb{N}\right\}$. This is an increasing sequence; the infimum, which is also a minimum, is $\frac{1}{4}$ and the supremum is $\frac{1}{3}$.

(d) $\left\{\frac{m}{m+n} : m, n \in \mathbb{N}\right\}$. This is the same set as part (a)!

Section 1.4

1.4.5

We claim that if a and b are real numbers with $a < b$, there is an irrational number t such that $a < t < b$. Consider the real numbers $a - \sqrt{2} < b - \sqrt{2}$. By the density of the rationals in the reals, there is a rational r such that $a - \sqrt{2} < r < b - \sqrt{2}$. Adding $\sqrt{2}$ to all three terms we see that $a < r + \sqrt{2} < b$. Now by Exercise 1.4.1 from workshop, the sum of a rational number and an irrational number is always irrational, so in particular $r + \sqrt{2}$ is irrational. Therefore we are done.

Other Problems

Problem 6

Let S be nonempty and bounded below, say by ℓ. Then for any $s \in S$, we have $\ell \leq s$, implying that $-s \leq ell$. So ℓ is an upper bound for $-S$. Since $-S$ is bounded above it has a supremum by the Axiom of Completeness, call it a. We claim that $-a$ is the infimum of S. First, since a is an upper bound of $-S$, we have $-s \leq a$ for all $-s \in -S$, so we see that $-a \leq s$ for all $s \in S$, hence $-a$ is a lower bound for S. Furthermore suppose that ℓ is an arbitrary lower bound for S, then as we argued above $-\ell$ is an upper bound for $-S$, so $a \leq -\ell$, implying that $\ell \leq -a$. Hence a is greater than or equal to any lower bound for S. Ergo a is the infimum of S. We conclude that bounded below nonempty subsets of the real numbers have infima.