Homework 13 Solutions

April 19, 2021

Section 5.2

Problem 5.2.2

(a) Possible. Let $f(x) = x^{\frac{1}{3}}$ and $g(x) = x^{\frac{2}{3}}$, such that neither f nor g is differentiable at 0 but $fg(x) = x$ is.

(b) Possible. Let $g(x) \equiv 0$ be the constant zero function and $f(x)$ be any function not differentiable at 0. Then $fg(x) \equiv 0$ is differentiable at 0. [Note that if we added the hypothesis that $g(0)$ and $g'(0)$ are nonzero, this would become impossible – why?]

(c) Impossible. Suppose g and $f + g$ are both differentiable at zero. Then $f = (f + g) - g$ is also differentiable at 0 by the Algebraic Differentiability Theorem.

(d) Possible. Consider

$$f(x) = \begin{cases} x^2 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

which we saw in class is differentiable only at $c = 0$.

0.1 Problem 5.2.9

(a) True; derivatives have the intermediate value property and therefore if the derivative of f is defined on an interval A and not constant, its image $f'(A)$ is a nondegenerate interval (that is, not a single point), and therefore contains irrational values.

(b) False. Consider the example

$$f(x) = \begin{cases} 2x^2 \sin\left(\frac{1}{x}\right) + x & x > 0 \\ x & x \leq 0 \end{cases}$$

whose derivative is

$$f'(x) = \begin{cases} 4x \sin\left(\frac{1}{x}\right) - 2\cos\left(\frac{1}{x}\right) + 1 & x > 0 \\ 1 & x \leq 0 \end{cases}$$

We see that $f'(0) = 1 > 0$. However, for any $\epsilon > 0$, choose n a natural number large enough that $x = \frac{1}{2\pi n} < \epsilon$. Then $f'(x) = 0 - 2 + 1 = -1$. So there is no interval around 0 on which $f'(x)$ is nonzero.

(c) True. For suppose not, then we have $f'(0) = a$ and $\lim_{x\to 0} f'(x) = L$ for $L \neq a$. Let $\epsilon = \frac{1}{2}|a - L|$, and choose δ such that $0 < |x - 0| < \delta$ implies $|f'(x) - L| < \epsilon$. So on the interval $(-\delta, \delta)$, the image of f' lies entirely inside the interval $(L - \epsilon, L + \epsilon)$ except that it contains the point a which is outside this interval. This is a contradiction, since derivatives have the intermediate value property, which implies that the image of an interval under the derivative function is an interval.

Section 5.3

Problem 5.3.2

Suppose f is differentiable on an interval A. Then f is also continuous on A. Suppose that f is not one-to-one on A, that is, suppose there is some $x < y$ such that $f(x) = f(y)$. Then by the Mean Value Theorem applied to $[x, y]$, there is some $c \in (x, y)$ such that

$$f'(c) = \frac{f(y) - f(x)}{x - y} = \frac{0}{x - y} = 0$$

This implies that $f'(c) = 0$. So, if $f'(c) \neq 0$ on all of A, f must be one-to-one on A.

The converse is not true by considering the example of $f(x) = x^3$, which is one-to-one on $\mathbb{R}$ but has $f'(0) = 0$.

Problem 5.3.3

We have h a differentiable, and therefore also continuous, function on $[0, 3]$ such that $h(0) = 1$, $h(1) = 2$, and $h(3) = 2$.

(a) Consider the function $f(x) = h(x) - x$. We observe that $f(0) = 1 - 0 = 1 > 0$ and $f(3) = 2 - 3 = -1 < 0$. By the Intermediate Value Theorem, we see there is a point d in $(0, 3)$ at which $f(d) = 0$, which means that $h(d) = d$.

(b) By the Mean Value Theorem, there is some $c \in (0, 3)$ with the property that

$$h'(c) = \frac{h(3) - h(0)}{3 - 0} = \frac{2 - 1}{3} = \frac{1}{3}.$$

(c) By Rolle's Theorem, since $h(1) = h(3)$, there is some $e \in (1, 3)$ with the property that $h'(e) = 0$. Now, recall that derivatives have the intermediate value property; the image of an interval under the derivative is an interval. We have seen that $h'([0, 3])$ contains 0 and $\frac{1}{3}$; ergo, it contains all values between them, including $\frac{1}{4}$. So there is some point x in the domain for which $h'(x) = \frac{1}{4}$.

Problem 5.3.6

(a) We have g differentiable on all of $[0, a]$, and therefore also continuous on $[0, a]$, with $g(0) = 0$ and $|g'(x)| \leq M$ for all $x \in [0, a]$. The claim is clearly true for $x = 0$. For any $x \in (0, a]$, the Mean Value Theorem tells us that there exists a $c \in (0, x)$ such that

$$g'(c) = \frac{g(x) - g(0)}{x - 0} = \frac{g(x)}{x}.$$

Since $|g'(c)| \le M$, we conclude that $\left|\frac{g(x)}{x}\right| \le M$, or equivalently $|g(x)| \le Mx$. As x was arbitrary we are done.

(b) We have h twice-differentiable on $[0, a]$, which implies that h and h' are continuous on $[0, a]$, and furthermore $h(0) = h'(0) = 0$ and $|h''(x)| \le M$ for all $x \in [0, a]$. Applying part (a) of this problem to h' we conclude that $|h'(x)| \le Mx$ for all $x \in [0, a]$.
Now we let $g(x) = \frac{x^2}{2}$ and apply the Generalized Mean Value Theorem to $h(x)$ and $g(x)$. For $x \in (0, a]$, the theorem tells us there is some point $c \in (0, x)$ such that

$$h'(c)[g(x) - g(0)] = g'(c)[h(x) - h(0)]$$

or

$$h'(c) \cdot \frac{x^2}{2} = x \cdot h(x)$$

Since we have $|h'(c)| \le Mc < Mx$, we see that

$$|h(x)| \le (Mx) \cdot \left(\frac{x^2}{2}\right) \cdot \frac{1}{x} = \frac{Mx^2}{2}$$

as desired. Since the claim is also clearly true for $x = 0$ we are done.

(c) Claim: Let $f : [0, a] \to \mathbb{R}$ be thrice differentiable with the property that $f(0) = f'(0) = f''(0) = 0$ and $|f'''(x)| \le M$ on $[0, a]$. Then $|f(x)| \le \frac{Mx^3}{6}$ for all $x \in [0, a]$.

Proof: Applying part (b) of this problem to f' we conclude that $|f'(x)| \le \frac{Mx^2}{2}$ on $[0, a]$. We now let $g(x) = \frac{x^3}{6}$. For any $x \in (0, a]$, apply the Generalized Mean Value Theorem to g and f to find $c \in (0, x)$ such that

$$f'(c)[g(x) - g(0)] = g'(c)[f(x) - f(0)]$$

which after subbing in the values we know becomes

$$f'(c) \cdot \frac{x^3}{6} = \frac{x^2}{2} \cdot f(x)$$

Since $|f'(c)| \le \frac{Mc^2}{2} \le \frac{Mx^2}{2}$, we have that

$$|f(x)| \le \frac{Mx^2}{2} \cdot \frac{x^3}{6} \cdot \frac{2}{x^2}$$

which gives $|f(x)| \le \frac{Mx^3}{6}$ as desired. Since the claim is also clearly true for $x = 0$ we are done.

Problem 5.3.9

Assume that f and g are continuous on an interval containing a, the derivatives f' and g' exist on the interval exist on the interval and are continuous at a, and $g'(a) \neq 0$. Notice that in this situation $\lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)}$. Suppose that $f(a) = g(a) = 0$. Then we have

$$\begin{aligned} f(x) &= f'(a)(x - a) + \epsilon_1(x)(x - a) \\ g(x) &= g'(a)(x - a) + \epsilon_2(x)(x - a) \end{aligned}$$

where $\epsilon_i(a) = 0$ and ϵ_i is continuous at a for $i = 1, 2$. So we have that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(a)(x - a) + \epsilon_1(x)(x - a)}{g'(a)(x - a) + \epsilon_2(x)(x - a)} = \lim_{x \to a} \frac{f'(a) + \epsilon_1(x)}{g'(a) + \epsilon_2(x)} = \frac{f'(a)}{g'(a)}.$$

Other Problems

0.2 Problem 5

Let $y < x$ be two real numbers. Since $f(x) = \cos x$ is continuous and differentiable with derivative $f'(x) = -\sin x$ everywhere, the Mean Value Theorem tells us there is some $c \in (y, x)$ with the property that

$$-\sin c = \frac{\cos x - \cos y}{x - y}$$

But $|-\sin c| \leq 1$. So

$$1 \geq \frac{|\cos x - \cos y|}{|x - y|}$$

implying that $|x - y| \geq |\cos x - \cos y|$.

0.3 Problem 6

(a) To evaluate $\lim_{x\to 0} \frac{1-\cos x}{x^2}$, we observe that the numerator and denominator are both differentiable on a neighborhood of 0 and both evaluate to 0 at 0. So we may apply L'Hospital's Rule to see that

$$\lim_{x\to 0} \frac{1 - \cos x}{x^2} = \lim_{x\to 0} \frac{\sin x}{2x}$$

if we can show that the limit on the right exists. However, we see that $\frac{\sin x}{2x}$ still has the property that both the numerator and denominator are differentiable on a neighborhood of 0 and both evaluate to 0 at 0. Ergo we apply L'Hospital's Rule again, seeing that

$$\lim_{x\to 0} \frac{\sin x}{2x} = \lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2}$$

We conclude that $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$.

(b) To evaluate $\lim_{x\to 0} \left[\frac{1}{\sin x} - \frac{1}{x}\right]$, we first rewrite the limit as

$$\lim_{x\to 0} \frac{x - \sin x}{x \sin x}$$

We use two successive applications of L'Hospital's Rule, noting at each step that the numerator and denominator are both differentiable on a neighborhood of 0 and evaluate to 0 at 0, to observe that

$$\lim_{x\to 0} \frac{x - \sin x}{x \sin x} = \lim_{x\to 0} \frac{1 - \cos x}{\sin x + x \cos x} = \lim_{x\to 0} \frac{\sin x}{2\cos x - x \sin x} = \frac{0}{2 - 0} = 0.$$

(c) To evaluate $\lim_{x\to 0}(1+2x)^{\frac{1}{x}}$, we in fact start by computing $\lim_{x\to 0} \ln[(1+2x)^{\frac{1}{x}}]$. We observe that

$$\ln[(1 + 2x)^{\frac{1}{x}}] = \frac{1}{x} \cdot \ln(1 + 2x) = \frac{\ln(1 + 2x)}{x}.$$

Since $\ln(1 + 2x)$ and x are both differentiable on a neighborhood of 1 and both evaluate to 0 at 0, we apply L'Hospital's Rule to see that

$$\lim_{x\to 0} \frac{\ln(1 + 2x)}{x} = \lim_{x\to 0} \frac{\frac{2}{1+2x}}{1} = 2.$$

Now we finish the problem. Since the function e^x is continuous on $\mathbb{R}$, we have that

$$\begin{aligned}\lim_{x\to 0}(1+2x)^{\frac{1}{x}} &= \lim_{x\to 0} e^{\ln[(1+2x)^{\frac{1}{x}}]} \\ &= e^{\lim_{x\to 0}\ln[(1+2x)^{\frac{1}{x}}]} \\ &= e^2.\end{aligned}$$