Homework 12 Solutions

April 19, 2021

Section 4.4

Problem 4.4.2

(a) No, $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0,1)$, either by the $\epsilon - \delta$ argument given in class, or by observing that there is a Cauchy sequence $\left(\frac{1}{n}\right)$ in $(0,1)$ which is mapped to a sequence $\left(f\left(\frac{1}{n}\right)\right) = (n)$ which is not Cauchy.

(b) Yes, $g(x) = \sqrt{1+x^2}$ is uniformly continuous on $(0,1)$. Note that it can be continuously extended to a function $\widetilde{g}$ on $[0,1]$, for example by taking $\widetilde{g}(x) = \sqrt{1+x^2}$ on $[0,1]$.

(c) Yes, $h(x) = x\sin\left(\frac{1}{x}\right)$ is uniformly continuous on $(0,1)$. Note that it can be continuously extended to a function $\widetilde{h}$ on $[0,1]$, by taking $\widetilde{h}(1) = \sin(1)$ and $\widetilde{h}(0) = 0$.

Problem 4.4.5

Let g be defined on (a,c) such that g is uniformly continuous on $(a,b]$ and on $[b,c)$. Let $\epsilon > 0$, and pick δ_1 such that if $x, y \in (a,b]$ and $|x-y| < \delta_1$, then $|g(x)-g(y)| < \frac{\epsilon}{2}$, and likewise δ_2 such that if $x, y \in [b,c)$ and $|x-y| < \delta_2$ then $|g(x)-g(y)| < \frac{\epsilon}{2}$. Then let $x \leq y$ be any two elements of (a,c), and assume that $|x-y| < \delta = \min\{\delta_1, \delta_2\}$. If either $x \leq y \leq b$ or $b \leq x \leq y$, it follows immediately that $|f(x)-f(y)| < \frac{\epsilon}{2} < \epsilon$. The interesting case is when $x < b < y$. In that case we see that $b - x < y - x < \delta \leq \delta_1$, so since $x, b \in (a,b]$, we have $|f(x)-f(b)| < \frac{\epsilon}{2}$. Likewise since $y - b < y - x < \delta < \delta_2$, so since $y, b \in [b,c)$, we have that $|f(b)-f(y)| < \frac{\epsilon}{2}$. Ergo $|x-y| < \delta$ and $x, y \in (a,c)$ implies that $|f(x)-f(y)| \leq |f(x)-f(b)| + |f(b)-f(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon$.

Problem 4.4.6

(a) Possible. Let $f : (0,1) \to \mathbb{R}$ be $f(x) = \frac{1}{x}$. Then let $(x_n) = \left(\frac{1}{n}\right)$. This is a Cauchy sequence, but $(f(x_n)) = (n)$ is not.

(b) Impossible; as proved in class, the image of a Cauchy sequence in a domain A under a function $f : A \to \mathbb{R}$ which is uniformly continuous is always a Cauchy sequence.

(c) Impossible. Suppose $f : [0,\infty) \to \mathbb{R}$. Let (x_n) be a Cauchy sequence in $[0,\infty)$. Then (x_n) is bounded, hence contained in some $[0,M]$. But f is uniformly continuous on $[0,M]$ because $[0,M]$ is compact, so $(f(x_n))$ is a Cauchy sequence.

Problem 4.4.10

(a) Let $f(x), g(x)$ be uniformly continuous on A. Let $\epsilon > 0$. Choose δ_1 such that for $x, y \in A$ we have that $|x-y| < \delta_1$ implies that $|f(x) - f(y)| < \frac{\epsilon}{2}$ and δ_2 such that $|x-y| < \delta_2$ implies that $|g(x) - g(y)| < \frac{\epsilon}{2}$. Then for $x, y \in A$, we see that $|x-y| < \delta = \min\{\delta_1, \delta_2\}$ implies that $|f + g(x) - f + g(y)| = |f(x) + g(x) - f(y) - g(y)| \leq |f(x) - f(y)| + |g(x) - g(y| < \epsilon$. So $f+g$ is uniformly continuous on A.

(b) It is not necessarily the case that if $f(x)$ and $g(x)$ are uniformly continuous on a domain A then their product is. Let $f(x) = x$ and $g(x) = x$ on $A = [0, \infty)$. Both of these functions are uniformly continuous on A because linear functions are uniformly continuous on any domain, but $f(x)g(x) = x^2$ is not uniformly continuous on $[0, \infty)$, by the proof given in class.

(c) It is not necessarily the case that if $f(x)$ and $g(x)$ are uniformly continuous on a domain A then their quotient is. To see this, consider $f(x) = 1$ and $g(x) = x$, both of which are uniformly continuous on $A = (0, 1)$ since linear functions are uniformly continuous on any domain, and observe that $\frac{f(x)}{g(x)} = \frac{1}{x}$ is not uniformly continous on $(0, 1)$.

(d) Let f be uniformly continuous on A and g be uniformly continuous on a domain B which contains $f(A)$. Let $\epsilon > 0$. Choose δ such that $x, y \in B$ and $|x-y| < \delta$ implies that $|g(x)-g(y)| < \epsilon$. Then choose δ' such that $x, y \in A$ and $|x-y| < \delta'$ implies that $|f(x) - f(y)| < \delta$. We then see that if $x, y \in A$ and $|x-y| < \delta'$, that implies that $f(x), f(y) \in B$ and $|f(x) - f(y)| < \delta$, which in turn implies that $|g(f(x)) - g(f(y))| < \epsilon$. As $\epsilon > 0$ was arbitrary, we conclude that $g \circ f$ is uniformly continuous on A.

Section 5.2

0.1 Problem 5.2.5

We set

$$f_a(x) = \begin{cases} x^a & x > 0 \\ 0 & x \leq 0 \end{cases}$$

(a) We claim that f_a is continuous at 0 if $a > 0$. To see this note that $f_a(0) = 0$, so f_a is continuous exactly when $\lim_{x\to 0} f_a(x) = 0$. Now recall that$\lim_{x\to 0} f_a(x) = 0$ is equivalent to the left-hand and right-hand limits of the function at $f_a(x)$ existing and both equalling 0. Since it is always the case that the left-hand limit $\lim_{x\to 0^-} f_a(x) = \lim_{x\to 0^-} 0 = 0$, it suffices to check that $\lim_{x\to 0^+} f_a(x)$ is zero. To the right of zero, $f_a(x) = x^a$, and

$$\lim_{x\to 0^+} x^a = \begin{cases} 0 & a > 0 \\ 1 & a = 0 \\ \infty & a < 0 \end{cases}$$

We conclude that f_a is continuous at 0 if $a > 0$.

(b) We claim that f_a is differentiable at 0 if $a > 1$. For the derivative exists if

$$\lim_{x\to 0} \frac{f_a(x) - f_a(0)}{x - 0} = \lim_{x\to 0} \frac{f_a(x)}{x}$$

exists. Again, this limit exists if the left-hand and right-hand limits are equal, and $\lim_{x\to 0^-} \frac{f_a(x)}{x} = \lim_{x\to 0^-} \frac{0}{x} = 0$. Moreover we see that $\lim_{x\to 0^+} \frac{f_a(x)}{x} = \lim_{x\to 0^+} x^{a-1}$ which by the same logic as in part (a) is equal to 0 exactly when $a - 1 > 0$, or when $a > 1$. Then the full derivative function is

$$f_a'(x) = \begin{cases} ax^{a-1} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

which is continuous at 0 since $a - 1 > 0$.

(c) A close variation on the argument above shows that f_a is twice-differentiable at 0 when $a > 2$, and so on.

Other Problems

Problem 5

(a) We want to compute the derivative of $f(x) = \frac{3x+4}{2x-1}$ at $x = 1$. We see that

$$\begin{aligned} f'(1) &= \lim_{x\to 1} \frac{f(x) - f(1)}{x - 1} & &= \lim_{x\to 1} \frac{\frac{3x+4}{2x-1} - 7}{x - 1} \\ &= \lim_{x\to 1} \frac{-11x + 11}{x - 1} \\ &= \lim_{x\to 1} \frac{-11(x-1)}{x - 1} \\ &= \lim_{x\to 1} -11 & &= -11 \end{aligned}$$

(b) We want to compute the derivative of $g(x) = x^2 \cos x$ at $x = 0$. We see that

$$\begin{aligned} g'(0) &= \lim_{x\to 0} \frac{g(x) - g(0)}{x - 0} \\ &= \lim_{x\to 0} \frac{x^2 \cos x - 0}{x - 0} \\ &= \lim_{x\to 0} \frac{x^2 \cos x}{x} \\ &= \lim_{x\to 0} x \cos x \\ &= 0 \end{aligned}$$

(c) We want to compute the derivative of $h(x) = \frac{1}{x}$ at any $c \neq 0$. We see that

$$\begin{aligned} h'(c) &= \lim_{x \to c} \frac{h(x) - h(c)}{x - c} \\ &= \lim_{x \to c} \frac{\frac{1}{x} - \frac{1}{c}}{x - c} \\ &= \lim_{x \to c} \frac{\frac{c-x}{cx}}{x - c} \\ &= \lim_{x \to c} \frac{-1}{cx} \\ &= -\frac{1}{c^2} \end{aligned}$$