Homework 11 Solutions

April 4, 2021

Section 4.4

Problem 4.4.8

(a) Impossible; the image of a compact set under a continuous function is compact.

(b) Possible; let

$$f(x) = \begin{cases} 0 & 0 < x < \frac{1}{4} \\ 4x - 1 & \frac{1}{4} < x < \frac{1}{2} \\ 1 & \frac{1}{2} < x \end{cases}$$

(c) Let

$$g(x) = \frac{\left|\sin\left(\frac{1}{x}\right)\right| + x}{1 + 2x}$$

on $[0, 1)$. We observe that because all four terms in the expression are positive, this is always a positive number; moreover, since $\left|\sin\left(\frac{1}{x}\right)\right| + x \leq 1 + x < 1 + 2x$, we see that $g(x) < 1$. Since the image of an interval is an interval, to show that $g((0, 1])$ is $(0, 1)$ it suffices to check that we can find an x such that $g(x) < \epsilon$ for all $\epsilon > 0$ and a y such that $g(y) > 1 - \epsilon$ or equivalently $1 - g(y) < \epsilon$ likewise for all $\epsilon > 0$.

So, let $\epsilon > 0$. Then pick $x = \frac{1}{2\pi n} < \epsilon$. We have $g(x) = \frac{0+x}{1+2x} < x < \epsilon$. Similarly if we pick y such that $y = \frac{1}{2\pi n + \frac{\pi}{2}} < \epsilon$, we have that $1 - g(y) = 1 - \frac{1+y}{1+2y} = \frac{y}{1+2y} < y < \epsilon$. So $g((0, 1]) = (0, 1)$.

Problem 4.4.12

(a) False. Let $f : \mathbb{R} \to \mathbb{R}$ be the constant function $f(x) = 0$ for all $x \in \mathbb{R}$. Certainly f is continuous on $\mathbb{R}$. Then $\{0\}$ is finite but $f^{-1}(\{0\}) = \mathbb{R}$ is not.

(b) False, by the same example as (a); $\{0\}$ is compact but $\mathbb{R}$ is not.

(c) False, again by the same example.

(d) True, by taking complements of the characterization of continuity in terms of preimages of open sets (also stated in Exercise 4.4.11).

Problem 4.5.2

(a) Possible; let $f : (0,1) \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} \frac{1}{4} & 0 < x < \frac{1}{4} \\ x & \frac{1}{4} \le x \le \frac{3}{4} \\ \frac{3}{4} & \frac{3}{4} < x < 1 \end{cases}$$

so that $f((0,1)) = \left[\frac{1}{4}, \frac{3}{4}\right]$.

(b) Impossible; a closed interval is compact, and the image of a compact set under a continuous function is compact, hence in particular closed.

(c) Possible; consider $f : (0,\infty) \to \mathbb{R}$ given by $f(x) = (x-1)^2$, so that $f((0,\infty)) = [0,\infty)$.

(d) Impossible; $\mathbb{R}$ is connected and $\mathbb{Q}$ is not, but the image of a connected set under a continuous function is always connected.

Other Problems

Problem 6

(a) Let $f(x) = \cos x - x$. We observe that f is continuous on $\left[0, \frac{\pi}{2}\right]$ and $f(0) = 1$ whereas $f\left(\frac{\pi}{2}\right) = -\frac{\pi}{2}$. We conclude by the Intermediate Value Theorem that there is some $x \in \left(0, \frac{\pi}{2}\right)$ with the property that $f(x) = 0$, or equivalently $x = \cos x$.

(b) Let $g(x) = xe^x - 2$. We observe that f is continuous on $[0,1]$ and $g(0) = -2$ whereas $g(1) = e - 2 > 0$. We conclude by the Intermediate Value Theorem that there is some $x \in (0,1)$ with the property that $g(x) = 0$, or $xe^x = 2$.

Problem 7

Let $p(x) = a_n x^n + \cdots + a_1 x + a_0$ be a polynomial of odd degree. Note that $p(x)$ is continuous on $\mathbb{R}$. Without loss of generality let $a_n > 0$, since we could multiply $p(x)$ by -1 without changing its roots. Choose $y = \max\{|a_{n-1}|, \ldots, |a_0|\}$ and $x > \max\{1, \frac{ny}{a_n}\}$ so that

$$\begin{aligned} p(x) &= a_n x^n + \cdots + a_1 x + a_0 \\ &> a_n x^n - |a_{n-1}|x^{n-1} - \cdots - |a_1|x - |a_0| \\ &> ny(x^{n-1}) - |a_{n-1}|x^{n-1} - \cdots - |a_1|x^{n-1} - |a_0|x^{n-1} \\ &> nyx^{n-1} - nyx^{n-1} \\ &= 0 \end{aligned}$$

Similarly choose $z < 0$ so that $p(z) < 0$. Then there is some $r \in (z, x)$ with the property that $p(r) = 0$ by the Intermediate Value Theorem.