Math 311: Final

April 29, 2021

Instructions

You have three hours to take the exam. There are nine questions, each of which is worth five points. You should not use any notes, books, websites, or other aids. After time is called, please upload your solutions, after which you will be asked to record a brief video of yourself explaining one of your solutions for authentication purposes.

Problem 1

For each of the following, either give an example of the object described (no need to justify your answer) or explain why it is impossible to do so.

(a) A countable subset of $[-1, 1]$ with no limit points.

Impossible; if we list the elements $(x_1, x_2, \dots)$ of our set A, then this is a bounded sequence, so by Bolzano-Weierstrass it has a convergent subsequence $x_{n_k} \to x$. As x appears at most once in (x_{n_k}), after possibly deleting an instance of x from the subsequence we conclude that x is a limit point of A.

(b) A Cauchy sequence (x_n) in a set A and a continuous function $f : A \to \mathbb{R}$ such that $(f(x_n))$ is not Cauchy.

Possible; consider $f : (0, 1) \to \mathbb{R}$ given by $f(x) = \frac{1}{x}$ and $(x_n) = \left(\frac{1}{n}\right)$.

(c) Two functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ with the property that neither f nor g is differentiable at 0, but fg is differentiable at 0.

Possible; consider $f(x) = x^{\frac{1}{3}}$ and $g(x) = x^{\frac{2}{3}}$.

(d) A continuous function $f : \mathbb{R} \to \mathbb{R}$ with the property that $f(\mathbb{R}) = \mathbb{Q}$.

Impossible; $\mathbb{R}$ is connected and $\mathbb{Q}$ is not.

Problem 2

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and let $E \subset \mathbb{R}$. Prove that $f(\overline{E}) \subseteq \overline{f(E)}$.

Since the closure of any set is closed and the preimage of a closed set under a continuous function is closed, we have that $f^{-1}(\overline{f(E)})$ is closed. Moreover, since if $x \in E$, we have $f(x) \subset f(E) \subset \overline{f(E)}$, so $x \in f^{-1}(\overline{f(E)})$. Hence $E \subset f^{-1}(\overline{f(E)})$. But $\overline{E}$ is the smallest closed set containing E, so this implies that $\overline{E} \subset f^{-1}(\overline{f(E)})$. Ergo $f(\overline{E}) \subseteq \overline{f(E)}$. A direct argument involving checking that any limit point of E is sent to either a point of $f(E)$ or a limit point of $f(E)$ is also possible.

Problem 3

Suppose f is differentiable on $\mathbb{R}$ and $f'(x) \leq 4$ for all $x \in \mathbb{R}$. Prove that there is at most one value $a > 2$ such that $f(a) = a^2$.

Since f is differentiable on $\mathbb{R}$, it follows that f is also continuous on $\mathbb{R}$. Suppose there are two values $a < b$ such that $f(a) = a^2$ and $f(b) = b^2$. By the Mean Value Theorem, there would then be a point $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Substituting in we have

$$\frac{b^2 - a^2}{b - a} = f'(c) \leq 4$$

so we see that in this case $b + a \leq 4$. Hence it is not the case that both a and b are greater than 2.

Problem 4

Suppose that f is a differentiable function on an interval A with the property that $|f'(x)| \leq M$ on A. Prove that f is uniformly continuous on A.

Let $\epsilon > 0$, and let $\delta = \frac{\epsilon}{M}$. Suppose that $x, y \in A$ such that $|x - y| < \delta$. Then by the Mean Value Theorem there is some $c \in (x, y)$ such that

$$\frac{f(x) - f(y)}{x - y} = f'(c)$$

Taking the absolute value of both sides we see that $\frac{|f(x)-f(y)|}{|x-y|} = |f'(c)| \leq M$, or $|f(x) - f(y)| \leq M|x - y| < \epsilon$. Since $\epsilon > 0$ was arbitrary we are done.

Problem 5

Compute the derivative functions of the following functions, where they exist.

(a) $g(x) = xe^{|x|}$
We may rewrite this function as

$$g(x) = \begin{cases} xe^{-x} & x < 0 \\ xe^{x} & x > 0 \end{cases}$$

The interesting case is when $x = 0$. We see that the lefthand limit is $\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{xe^{-x}}{x} = \lim_{x\to 0^-} e^{-x} = 1$ whereas the righthand limit is $\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{xe^{x}}{x} = \lim_{x\to 0^+} = 1$. We conclude that

$$g'(x) = \begin{cases} e^{-x} - xe^{-x} & x < 0 \\ 1 & x = 0 \\ e^{x} + xe^{x} & x > 0 \end{cases}$$

which we may repackage as $g'(x) = (1 + |x|)e^{|x|}$.

(b)

$$f(x) = \begin{cases} x \sin x & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Let c be any real number. Recall that there is a sequence of rational numbers (x_n) such that $x_n \to c$ and a sequence of irrational numbers (y_n) such that $y_n \to c$. We observe that $f(x_n) = x_n \sin(x_n) \to c \sin c$ and $f(y_n) = 0 \to 0$, so f is discontinuous for any c such that $c \sin(c) \neq 0$. Since differentiability implies continuity we need only investigate points where $c \sin c = 0$, or where $c = n\pi$ for some integer n.

For $c = n\pi$, the derivative at c if it is exists is $\lim_{x\to c} \frac{f(x)-c\sin(c)}{x-c}$. If $n \neq 0$, if we approach along the sequence (x_n) this quotient is $\frac{x_n \sin(x_n)-c\sin(c)}{x-c} = \sin(c) + c\cos(c) = c$ and if we approach along the sequence (y_n) this quotient is $\frac{0-0}{x-c} = 0 \to 0$. So, f is not differentiable at $c = n\pi$ for $n \neq 0$.

However, at $c = 0$, the derivative is $\lim_{x\to 0} \frac{f(x)}{x}$ if it exists, and $0 \leq \left|\frac{f(x)}{x}\right| \leq |\sin x|$, so by the Squeeze Theorem for Functional Limits we conclude that $f'(c) = 0$.

Problem 6

Let f be an increasing function on (a, b). Prove that for every $c \in (a, b)$, the left-hand limit $\lim_{x\to c^-} f(x)$ exists and is equal to $\sup\{f(x) : x \in (a, c)\}$. [Similarly with the righthand limit and the infimum, but you don't have to prove this.]

Let $\alpha = \sup\{f(x) : x \in (a, c)\}$. For every $\epsilon > 0$, there is some $x_0 \in (a, c)$ such that $f(x_0) > \alpha - \epsilon$. Now, if we have x such that $x_0 < x < c$, we see that $f(x_0) \leq f(x) \leq \alpha$. So in particular if $\delta = c - x_0$, we have that $c - delta < x < c$ implies that $\alpha - \epsilon < f(x) < \alpha$. As $\epsilon > 0$ was arbitary, we have that $\lim_{x\to c^-} f(x) = \alpha$ as desired.

Problem 7

Compute the following limits.

(a) $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}$
We multiply both the numerator and denominator by $\sqrt{1+x}+\sqrt{1-x}$, obtaining

$$\begin{aligned}\lim_{x\to 0} \frac{(1+x)-(1-x)}{x(\sqrt{1+x}+\sqrt{1-x})} &= \lim_{x\to 0} \frac{2x}{x(\sqrt{1+x}+\sqrt{1-x})} \\ &= \lim_{x\to 0} \frac{2}{(\sqrt{1+x}+\sqrt{1-x})} \\ &= \frac{2}{2} \\ &= 1\end{aligned}$$

(b) $\lim_{x\to 0^+} x^x$

We replace the limit with

$$\begin{aligned}\lim_{x\to 0^+} x^x &= \lim_{x\to 0^+} e^{\ln(x^x)} \\ &= \lim_{x\to 0^+} e^{x\ln(x)}\end{aligned}$$

We observe that the term in the exponent has

$$\begin{aligned}\lim_{x\to 0^+} x\ln(x) &= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} \\ &= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \\ &= \lim_{x\to 0^+} -x \\ &= 0\end{aligned}$$

where the second equality is by L'Hospital's Rule since the limit on the right exists. So since e^x is a continuous function, the original limit is $e^0 = 1$.

(c) $\lim_{x\to 0} \frac{1-\cos x}{e^x-1}$

We observe that the numerator and denominator evaluate to 0 at 0, so the limit is equal to

$$\lim_{x\to 0} \frac{\sin x}{e^x} = \frac{0}{1} = 0$$

by L'Hospital's Rule since the latter exists.

Problem 8

Consider the functions $p_n(x) = 1 + x + x^2 + \cdots + x^n$.

(a) Prove that $p_n(x)$ is uniformly continuous on $(-1, 1)$.

We see that $p_n(x)$ is continuous on the compact set $[-1, 1]$, hence uniformly continuous on $[-1, 1]$, hence also uniformly continuous on the subset $(-1, 1)$.

(b) Let f be the function defined by $f(x) = \lim_{n\to\infty} p_n(x)$ on $(-1,1)$. Give a simple algebraic expression for f. [Hint: The polynomials p_n are partial sums of $\sum_{k=0}^{\infty} x^k$.]

We see that $f(x) = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ on $(-1,1)$.

(c) Is f uniformly continuous on $(-1,1)$?

No. For example, we see that the Cauchy sequence $(x_n) = \left(1 - \frac{1}{n}\right)$ in $(-1,1)$ is mapped to $(f(x_n)) = (n)$, which is not Cauchy.

Problem 9

Prove the following statements.

(a) Let $f(x)$ be twice differentiable on an open interval A with $f'' \equiv 0$ on the interval. Prove that $f(x)$ must be of the form $f(x) = ax + b$ for some $a, b \in \mathbb{R}$ on A.

We observe that $f''(x) = 0$ on A implies that $f'(x) = a$ is a constant function on A. But since the constant function ax has derivative the constant function a, the complete set of functions with constant derivative a is exactly $f(x) = ax + b$ where b is another constant.

(b) There is some $x \in (1,\infty)$ for which $\ln x = 2 - x^2$.

Let $f(x) = \ln x + x^2 - 2$. We observe that $f(1) = 0 + 1 - 2 = -1$ and $f(e) = \ln(e) + e^2 - 2 = 1 + e^2 - 2 > 0$. So by the Intermediate Value Theorem there is some $x \in (1,e) \subseteq (1,\infty)$ for which $\ln x + x^2 - 2 = 0$, or equivalently $\ln x = 2 - x^2$.