Math 311: Sample Final

May 7, 2021

Instructions

You have three hours to take the exam. There are nine questions, each of which is worth five points. You should not use any notes, books, websites, or other aids. After time is called, please upload your solutions, after which you will be asked to record a brief video of yourself explaining one of your solutions for authentication purposes.

Problem 1

For each of the following, either give an example of the object described (no need to justify your answer) or explain why it is impossible to do so.

(a) A countable infinite connected subset of $\mathbb{R}$.

Impossible; nonempty connected subsets of $\mathbb{R}$ are intervals, and any interval which is not a single point contains uncountably many points.

(b) A bounded sequence with no subsequential limits.

Impossible; by Bolzano-Weierstrass, every bounded sequence has a convergent subsequence, and thus a subsequential limit.

(c) Two sets A and B in $\mathbb{R}$ such that $\sup A \leq \inf B$ and $A \cap B \neq \emptyset$.

Possible; consider $A = (0, 1]$ and $B = [1, 2)$.

(d) Two functions f and g uniformly continuous on a domain A such that $f(x)g(x)$ is not uniformly continuous on A.

Possible; consider $f(x) = g(x) = x$ on $[0, \infty)$. Both of these functions are uniformly continous on any domain, but $fg(x) = x^2$ is not uniformly continuous on $[0, \infty)$.

Problem 2

A function $f : \mathbb{R} \to \mathbb{R}$ is said to have a fixed point if $f(x) = x$. Prove that if f is differentiable on an interval A with $f'(x) \neq 1$, then f has at most one fixed point.

Since f is differentiable on A, we have that f is also continuous on A. Suppose that f has two fixed points $x < y$ in A, and apply the Mean Value Theorem to $[x, y]$. Then there is some $c \in (x, y)$ such that

$$f'(c) = \frac{f(y) - f(x)}{y - x} = \frac{y - x}{y - x} = 1$$

which contradicts the assumption that $f' \neq 1$ on A. So f has at most one fixed point.

Problem 3

Let f be uniformly continuous on a bounded set A. Prove that the image $f(A)$ is also bounded.

Suppose not, then there is a sequence of points (y_n) in $f(A)$ such that $y_n > n$ for all $n \in \mathbb{N}$. Now, since $y_n \in A$, there is some $x_n \in A$ such that $f(x_n) = y_n$. The sequence (x_n) is bounded, hence by Bolzano-Weierstrass it has a subsequence (x_{n_k}) which converges in $\mathbb{R}$ and in particular is Cauchy. But since f is uniformly continuous, the sequence $(f(x_{n_k})) = (y_{n_k})$ should be Cauchy as well, and in particular bounded. This is a contradiction since $y_{n_k} > n_k$ for all n_k.

Problem 4

We say that a continuous map $f : \mathbb{R} \to \mathbb{R}$ is open if for every open set $O \subset \mathbb{R}$, the image $f(O)$ is also an open set. Prove that an open map is strictly monotone. [Hint: For any $a < b$ in $\mathbb{R}$, where must the maximum and minimum values of f on $[a, b]$ lie?]

Let $a \in \mathbb{R}$, and consider any b such that $b > a$. We see that $f([a, b])$ is a compact connected set, hence either a closed interval or a point; since $f((a, b))$ is open, we in fact have that $f([a, b])$ is a closed interval $[c, d]$ and that c and d are not in the image $f((a, b))$, so they must be the image of a and b in some order. In particular, the maximum and minimum values of f on a closed interval occur at the endpoints.

We now have two cases. Suppose that $f(a) = c$ and $f(b) = d$, so that $f(a) < f(b)$. In this case, if $a < x < b$, then $f(x) \in (c, d)$, which implies that $f(a) < f(x)$, and if $a < b < y$, then the same argument shows that a and y map to the endpoints of an interval which contains $f(b)$, so a must also be the left-hand endpoint of that interval. Hence $f(a) < f(y)$. So we see that $f(a) < f(x)$ for all $x > a$. If we instead assumed that $f(a) = d$ and $f(b) = c$, we would have gotten $f(a) > f(x)$ for all $x > a$. So one of these things is true for every $a \in \mathbb{R}$.

Now, suppose we have an a in $\mathbb{R}$ with the property that $f(a) < f(x)$ for all $x > a$ and an a' in $\mathbb{R}$ with the property that $f(a) > f(x)$ for all $x > a$. Then suppose $a < a'$, and $x > a'$. Then we should have $f(a) < f(a')$ and $f(a) < f(x)$, but also we should have $f(a') > f(x)$. So, in particular, $f([a, x])$ does not take its maximum at either a or x, contradicting openness. Hence, we only get points of one type, so f is strictly monotone.

Remark: The roughly corresponding problem on the actual final is easier than this one.

Problem 5

Compute the derivative functions of the following functions where they exist.

(a) $f(x) = |x| + |x-1|$

We see that this function may be rewritten

$$f(x) = \begin{cases} 1-2x & x < 0 \\ 1 & 0 \le x \le 1 \\ 2x-1 & 1 < x \end{cases}$$

We conclude that

$$f(x) = \begin{cases} -2 & x < 0 \\ 0 & 0 < x < 1 \\ 2 & 1 < x \end{cases}$$

and fails to exist at 0 and 1.

(b)

$$g(x) = \begin{cases} \left(\sin^2 x\right) \cdot \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

The interesting case is $x = 0$. We see that

$$\begin{aligned} \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} &= \lim_{x \to 0^+} \frac{\sin^2(x) \sin\left(\frac{1}{x}\right)}{x} \\ &= \lim_{x \to 0^+} \left(\frac{\sin(x)}{x}\right) \cdot \sin x \cdot \sin\left(\frac{1}{x}\right). \end{aligned}$$

As $x \to 0$, we have that $\frac{\sin x}{x} \to 1$, the term $\sin x \to 0$, and $\sin\left(\frac{1}{x}\right)$ is bounded. So the entire limit is zero. Hence the derivative function is

$$g'(x) = \begin{cases} 2\sin x \cos x \sin\left(\frac{1}{x}\right) - \frac{\sin^2(x)}{x^2} \cos\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Problem 6

Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be two continuous functions with the property that $f(r) = g(r)$ for all $r \in \mathbb{Q}$. Prove that $f(x) = g(x)$ for all $x \in \mathbb{R}$.

Let x be any real number. Then there is a sequence of rational points (r_n) converging to x. By continuity, $f(r_n) \to f(x)$ and $g(r_n) \to g(x)$, but $(f(r_n))$ and $g(r_n))$ are the same sequence, so $f(x) = g(x)$.

Problem 7

Let $f(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}$, wherever this sum is a real number.

(a) For which values of x does this series converge, and to what?

We see that if $x = 0$ the series converges to 0. Otherwise, the series is geometric $\sum_{n=0}^{\infty} ar^n$ with $a = x^2$ and with ratio $r = \frac{1}{(1+x^2)^n}$, which has absolute value less than 1, so it converges to

$$\begin{aligned} f(x) &= \frac{a}{1-r} \\ &= \frac{x^2}{1 - \frac{1}{1+x^2}} \\ &= \frac{x^2}{\frac{x^2}{1+x^2}} \\ &= 1 + x^2 \end{aligned}$$

(b) At what points is the function $f(x)$ continuous?
We observe that the function

$$f(x) = \begin{cases} 1 + x^2 & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is continuous everywhere other than 0.

Problem 8

Compute the following limits.

(a) $\lim_{x\to 0} (\cos x)^{\frac{1}{x^2}}$
We replace the limit with

$$\lim_{x\to 0} e^{\ln((\cos x)^{\frac{1}{x^2}})}$$

which becomes

$$\lim_{x\to 0} e^{\frac{\ln(\cos x)}{x^2}}$$

The limit of the exponent is

$$\begin{aligned} \lim_{x\to 0} \frac{\ln(\cos x)}{x^2} &= \lim_{x\to 0} \frac{\frac{-\sin x}{\cos x}}{2x} \\ &= \lim_{x\to 0} -\frac{\sin x}{2x\cos x} \\ &= \lim_{x\to 0} -\frac{1}{2\cos x} \cdot \frac{\sin x}{x} \\ &= -\frac{1}{2}(1) \\ &= -\frac{1}{2} \end{aligned}$$

where the second step is an application of L'Hospital's Rule. So the entire limit is $e^{-\frac{1}{2}}$, or $\frac{1}{\sqrt{e}}$.

(b) $\lim_{x\to 0} \frac{\tan x - x}{x^3}$
We observe that the numerator and denominator both evaluate to zero at zero, so we know by L'Hospital's Rule that the limit above is equal to

$$\lim_{x\to 0} \frac{\sec^2(x) - 1}{3x^2}$$

if it exists. But the numerator and denominator are still continuous and evaluate to zero at zero, so this second limit is equal to

$$\lim_{x\to 0} \frac{2\sec^2(x)\tan x}{6x}$$

which after yet another application of the same principle becomes

$$\lim_{x\to 0} \frac{2\sec^2(x)\tan^2 x + 2\sec^4(x)}{6} = \frac{0+2}{6} = \frac{1}{3}.$$

(c) $\lim_{x\to 0} \frac{1}{e^x - 1} - \frac{1}{x}$

We rewrite the limit as

$$\lim_{x\to 0} \frac{x - e^x - 1}{xe^x}.$$

We observe that the numerator and denominator are both continuous and evaluate to zero at zero, so we know by L'Hospital's Rule that the limit above is equal to

$$\lim_{x\to 0} \frac{1 - e^x}{e^x + xe^x}$$

which by the same principle is equal to

$$\lim_{x\to 0} \frac{-e^x}{e^x + e^x + xe^x} = -\frac{1}{2}.$$

Problem 9

Prove the following statements.

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $f(a)f(b) < 0$ for two values $a < b \in \mathbb{R}$. Show that there is some $c \in (a, b)$ such that $f(c) = 0$.

Notice that $f(a)$ and $f(b)$ have the property that one is positive and one is negative. So, by the Intermediate Value Theorem, there must be a $c \in (a, b)$ such that $f(c) = 0$.

(b) Show that $x \leq \tan x$ for $x \in \left(0, \frac{\pi}{2}\right)$.
Let $f(x) = \tan x - x$. Observe that $f(0) = 0$ and $f'(x) = \sec^2(x) - 1$, which is positive on $\left[0, \frac{\pi}{2}\right)$. Hence f is increasing on this interval, so that $f(x) \geq 0$ on $\left(0, \frac{\pi}{2}\right)$, or equivalently $\tan x \geq x$.