Math 311: Sample Final

April 28, 2021

Instructions

You have three hours to take the exam. There are nine questions, each of which is worth five points. You should not use any notes, books, websites, or other aids. After time is called, please upload your solutions, after which you will be asked to record a brief video of yourself explaining one of your solutions for authentication purposes.

Problem 1

For each of the following, either give an example of the object described (no need to justify your answer) or explain why it is impossible to do so.

(a) A countable infinite connected subset of $\mathbb{R}$.

(b) A bounded sequence with no subsequential limits.

(c) Two sets A and B in $\mathbb{R}$ such that $\sup A \leq \inf B$ and $A \cap B \neq \emptyset$.

(d) Two functions f and g uniformly continuous on a domain A such that $f(x)g(x)$ is not uniformly continuous on A.

Problem 2

A function $f : \mathbb{R} \to \mathbb{R}$ is said to have a fixed point if $f(x) = x$. Prove that if f is differentiable on an interval A with $f'(x) \neq 1$, then f has at most one fixed point.

Problem 3

Let f be uniformly continuous on a bounded set A. Prove that the image $f(A)$ is also bounded.

Problem 4

We say that a continuous map $f : \mathbb{R} \to \mathbb{R}$ is open if for every open set $O \subset \mathbb{R}$, the image $f(O)$ is also an open set. Prove that an open map is strictly monotone. [Hint: For any $a < b$ in $\mathbb{R}$, where must the maximum and minimum values of f on $[a, b]$ lie?]

Problem 5

Compute the derivative functions of the following functions where they exist.

(a) $f(x) = |x| + |x-1|$

(b)

$$g(x) = \begin{cases} \left(\sin^2 x\right) \cdot \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Problem 6

Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be two continuous functions with the property that $f(r) = g(r)$ for all $r \in \mathbb{Q}$. Prove that $f(x) = g(x)$ for all $x \in \mathbb{R}$.

Problem 7

Let $f(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}$, wherever this sum is a real number.

(a) For which values of x does this series converge, and to what?

(b) At what points is the function $f(x)$ continuous?

Problem 8

Compute the following limits.

(a) $\lim_{x\to 0} (\cos x)^{\frac{1}{x^2}}$

(b) $\lim_{x\to 0} \frac{\tan x - x}{x^3}$

(c) $\lim_{x\to 0} \frac{1}{e^x - 1} - \frac{1}{x}$

Problem 9

Prove the following statements.

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $f(a)f(b) < 0$ for two values $a < b \in \mathbb{R}$. Show that there is some $c \in (a, b)$ such that $f(c) = 0$.

(b) Show that $x \leq \tan x$ for $x \in \left(0, \frac{\pi}{2}\right)$.