Math 311: Final

May 11, 2021

Instructions

You have three hours to take the exam. There are nine questions, each of which is worth five points. You should not use any notes, books, websites, or other aids. After time is called, please upload your solutions, after which you will be asked to record a brief video of yourself explaining one of your solutions for authentication purposes.

Problem 1

For each of the following, either give an example of the object described (no need to justify your answer) or explain why it is impossible to do so.

(a) A countable subset of $[-1, 1]$ with no limit points.

(b) A Cauchy sequence (x_n) in a set A and a continuous function $f : A \to \mathbb{R}$ such that $(f(x_n))$ is not Cauchy.

(c) Two functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ with the property that neither f nor g is differentiable at 0, but fg is differentiable at 0.

(d) A continuous function $f : \mathbb{R} \to \mathbb{R}$ with the property that $f(\mathbb{R}) = \mathbb{Q}$.

Problem 2

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and let $E \subset \mathbb{R}$. Prove that $f(\overline{E}) \subseteq \overline{f(E)}$.

Problem 3

Suppose f is differentiable on $\mathbb{R}$ and $f'(x) \leq 4$ for all $x \in \mathbb{R}$. Prove that there is at most one value $a > 2$ such that $f(a) = a^2$.

Problem 4

Suppose that f is a differentiable function on an interval A with the property that $|f'(x)| \leq M$ on A. Prove that f is uniformly continuous on A.

Problem 5

Compute the derivative functions of the following functions, where they exist.

(a) $g(x) = xe^{|x|}$

(b)

$$f(x) = \begin{cases} x \sin x & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Problem 6

Let f be an increasing function on (a, b). Prove that for every $c \in (a, b)$, the left-hand limit $\lim_{x \to c^-} f(x)$ exists and is equal to $\sup\{f(x) : x \in (a, c)\}$. [Similarly with the righthand limit and the infimum, but you don't have to prove this.]

Problem 7

Compute the following limits.

(a) $\lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}$

(b) $\lim_{x \to 0^+} x^x$

(c) $\lim_{x \to 0} \frac{1 - \cos x}{e^x - 1}$

Problem 8

Consider the functions $p_n(x) = 1 + x + x^2 + \cdots + x^n$.

(a) Prove that $p_n(x)$ is uniformly continuous on $(-1, 1)$.

(b) Let f be the function defined by $f(x) = \lim_{n \to \infty} p_n(x)$ on $(-1, 1)$. Give a simple algebraic expression for f. [Hint: The polynomials p_n are partial sums of $\sum_{k=0}^{\infty} x^k$.]

(c) Is f uniformly continuous on $(-1, 1)$?

Problem 9

Prove the following statements.

(a) Let $f(x)$ be twice differentiable on an open interval A with $f'' \equiv 0$ on the interval. Prove that $f(x)$ must be of the form $f(x) = ax + b$ for some $a, b \in \mathbb{R}$ on A.

(b) There is some $x \in (1, \infty)$ for which $\ln x = 2 - x^2$.