Math 227A: Suggested Exercises for Week 9

The following are suggested exercises for Week 9:


2. *The Splitting Principle* Let \( p: V \rightarrow B \) be a complex \( n \)-plane vector bundle, and \( P(V) \) its projectivization, with \( q: P(V) \rightarrow B \) the fibre bundle map. Show the vector bundle \( q^*V \) over \( P(V) \) always splits into the direct sum of a line bundle and an \((n-1)\)-plane bundle. Conclude that in proving natural relations for the Chern classes, it suffices to compute on a direct sum of line bundles.

3. Let \( V \rightarrow B \) be an \( n \)-plane complex bundle, and for \( m < n \) let \( \Lambda^m V \) be the vector bundle whose fibre is \( \Lambda^m F \) for every fibre \( F \) of \( B \). Compute the total Chern class of \( \Lambda^m(V) \) in terms of the Chern classes of \( V \).

4. Show that for \( M \) a smooth closed oriented manifold, the Euler class \( e(M) \) evaluated on the fundamental homology class of \( M \) is equal to the algebraic self-intersection number \( [TM]^2 \) of the zero-section of \( TM \) with itself. (Hint: think about Euler characteristic and vector fields.) Use this to show the adjunction formula: If \( S \) is a complex surface (so, a real 4-diml manifold) and \( C \) is an embedded complex curve of genus \( g(C) \), then \( 2g(C) - 2 = [C]^2 - c_1(S)[C] \), where \([C]^2\) is the algebraic self-intersection number of \( C \) inside \( S \) and \( c_1(S) \) is the Chern class of \( S \) evaluated on the homology class represented by \( C \).