Math 227A: Problem Set 4 and Suggested Exercises for Week 8

The following exercises are Problem Set 4, and are due on May 27:

1. Milnor and Stasheff 9B, 9C, 11D, 12B.

The following are suggested exercises for Week 6:

1. Milnor and Stasheff 8A, 9A, 12A.

2. Use last week’s exercises to give an example of a bundle with vanishing Euler class but no nowhere zero section.

3. A construction of the Steenrod squares

Everything below is in $\mathbb{Z}_2$-coefficients. Notation mostly chosen to match Hatcher 4L, although he does all the other Steenrod prime powers at the same time as the squares.

In theory, a squaring operation would involve $X \times X$, but it’s actually easier to work with $X \wedge X$, the smash product. This has a $\mathbb{Z}_2$ action generated by the map $T$ (for transposition) that interchanges the factors; the basepoint $x_0$ in the smash product is a fixed point of the action. Consider the Borel construction

$$\Gamma X = (X \wedge X) \times_{\mathbb{Z}_2} S^\infty := (((X \wedge X) \times S^\infty)/(x_1, x_2, z) \sim (x_2, x_1, -z)).$$

There is a fibre bundle $(X \wedge X) \hookrightarrow \Gamma X \xrightarrow{p} \mathbb{R}P^\infty$. Furthermore, since $x_0 \in X \times X$ is a fixed point, we have a basepoint section $\mathbb{R}P^\infty \hookrightarrow Y$. We let the quotient of $\Gamma X$ by this copy of $\mathbb{R}P^\infty$ be $\Lambda X$, which is now a basepointed space. If we restrict this entire construction to $S^1$, we get subspaces $\Gamma^1 X$ and $\Lambda^1 X$. (Exercise: All of these constructions are natural, and if $X$ has the structure of a CW complex, so do $\Gamma X$, $\Lambda X$, $\Gamma^1 X$, and $\Lambda^1 X$.)

Now, there is an isomorphism

$$H^*(X \wedge X) \rightarrow H^*(X) \otimes H^*(X).$$

(Exercise: Convince yourself this is true, if necessary.) In particular, we can think about $\alpha \otimes \alpha$ as an element of $H^{2n}(X \wedge X)$. Our goal is to construct an element $\lambda(\alpha) \in H^{2n}(\Lambda X)$ that restricts to $\alpha \otimes \alpha$ on each fibre $X \wedge X \subset \Lambda X$. By naturality, it suffices to construct a suitable $\lambda(\iota) \in H^{2n}(K(\mathbb{Z}_2, n))$, where $\iota$ is the fundamental class in $H^n(K(\mathbb{Z}_2, n))$. Give $K(\mathbb{Z}_2, n)$ a CW structure with $n$-skeleton the $n$-sphere. For notation purposes, elements $\alpha$ of $H^n(X)$ correspond to maps $\alpha: X \rightarrow K(\mathbb{Z}_2, n)$.

The main thing we need is that if $T$ is the transposition map on $K(\mathbb{Z}_2, n) \wedge K(\mathbb{Z}_2, n)$, the there is a basepoint-preserving homotopy between the maps $\iota \otimes \iota$ and $\iota \otimes \iota \circ T$ mapping $K(\mathbb{Z}_2, n) \wedge K(\mathbb{Z}_2, n) \rightarrow K(\mathbb{Z}_2, 2n)$. (Exercise: Construct this homotopy.) This is the same thing as saying that $T^*(\iota \otimes \iota) = \iota \otimes \iota$. The homotopy between these two maps induces a map $\Gamma^1(K(\mathbb{Z}_2, n)) \rightarrow K(\mathbb{Z}_2, 2n)$, which descends to a map $\Lambda^1 K(\mathbb{Z}_2, n) \rightarrow K(\mathbb{Z}_2, 2n)$. There is no obstruction to extending the resulting map to $\lambda: \Lambda K(\mathbb{Z}_2, n) \rightarrow K(\mathbb{Z}_2, 2n)$. (Exercise: Prove this.) The pullback of the canonical class in $H^{2n}(K(\mathbb{Z}_2, n))$ along $\lambda$ is the desired class $\lambda(\iota)$. From this and naturality we get $\lambda(\alpha)$ for arbitrary $\alpha$. 

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Finally, consider the inclusion $\mathbb{RP}^\infty \times X \hookrightarrow \Gamma(X)$ coming from the diagonal map $X \hookrightarrow X \times X$. We compose with the quotient map onto $\Lambda X$ to get a map $\nabla: \mathbb{RP}^\infty \times X \to \Lambda X$. Then we have

$$\nabla^*: H^*(\Lambda X) \to H^*(\mathbb{RP}^\infty) \otimes H^*(X)$$

For any $\alpha$ in $H^m(X)$, $\nabla^*(\lambda(\alpha)) = \sum_{i=0}^n h^{m-i} \otimes Sq^i(\alpha)$; that is, the Steenrod squares are whatever make this true.