Suggested Exercises for Week 7

The following are suggested exercises for Week 7:

1. Milnor & Stasheff 5A, 5B, 5E part 4, 4B, 4D, 6C, 6E. [Some of these are quite short.]

2. Here is a concrete construction of the Stiefel-Whitney classes. Given a real vector n-dimensional bundle $V \to B$, let $P(V) \to B$ be the projectivization of V; that is, $P(V)_b \simeq \mathbb{RP}^{n-1}$ is the space of lines in V_b for every $b \in B$. Apply the Leray-Hirsch theorem of last week's exercise sheet to the fibration $\mathbb{RP}^{n-1} \to P(V) \to B$ to show that $H^*(P(V);\mathbb{Z}_2) = H^*(B;\mathbb{Z}_2) \otimes H^*(\mathbb{RP}^{n-1};\mathbb{Z}_2)$. Let $1, h, \cdots, h^{n-1}$ be a basis for $H^*(\mathbb{RP}^{n-1};\mathbb{Z}_2)$. Then the class h^n, as an element of $H^*(P(V);\mathbb{Z}_2)$, can be written as $h^n = w_1 \otimes h^{n-1} + \cdots + w_{n-1} \otimes 1$ for some classes $w_i \in H^i(B;\mathbb{Z}_2)$. We claim these are the Stiefel-Whitney classes. (In the case of a line bundle, one avoids the trouble that $h = 0$ by first taking a Whitney sum with a trivial line bundle.) Prove this by showing they satisfy the axioms given in class.

3. Determine the reduced K-theory of the torus. [Hint: Consider suspensions.]

4. Show there is a three-dimensional real vector bundle over S^4 which is not the direct sum of a one-dimensional bundle and a two-dimensional bundle. [You're missing one fact you need, which is that an arbitrary two-dimensional vector bundle over S^4 can be given the structure of a complex line bundle. But otherwise you know enough to produce this example.]