Math 227A: Problem Set 3 and Suggested Exercises for Week 6

The following exercises are Problem Set 3, and are due on May 13:

1. Milnor and Stasheff 2A, 2B, 3A, 3E.

The following are suggested exercises for Week 6:

1. Show that the orthogonal complement of a subbundle is, up to isomorphism, independent of the choice of inner product.

2. Consider the diagonal embedding $\Delta : M \to M \times M$. Construct the tangent bundle and normal bundle of $\Delta \subset M \times M$, and show they are isomorphic.

3. Cohomology of fibre bundles Here is an outline for proving the Leray-Hirsch theorem: Let $F \to E \to B$ be a fibre bundle. Suppose that for some coefficient ring $R$, $H^*(F; R)$ is a finitely generated free $R$-module for all $n$ and there are classes $c_j \in H^*(E; R)$ whose restrictions form a basis for $H^*(F; R)$ for each fibre $F$. Then $H^*(E; R) \simeq H^*(F; R) \otimes H^*(B; R)$.

   - Start with a CW complex $B$. If $B$ is zero-dimensional this is trivial, so by induction, assume we already know the result for $(n-1)$-dimensional complexes. Let $B$ be $n$-dimensional, and $B'$ be the space obtained by deleting a single point $x_\alpha$ from the interior of each $n$-cell. Let $E' = p^{-1}(B')$. Show there is a commutative diagram of exact sequences

   $\egin{array}{ccccccccc}
   \cdots & \longrightarrow & H^*(B, B'; R) \otimes H^*(F; R) & \longrightarrow & H^*(B; R) \otimes H^*(F; R) & \longrightarrow & H^*(B'; R) \otimes H^*(F; R) & \cdots \\
   & & \downarrow & & \downarrow & & \downarrow & \\
   & \longrightarrow & H^*(E, E'; R) & \longrightarrow & H^*(E; R) & \longrightarrow & H^*(E'; R) & \\
   \end{array}$$

   - Show that the inclusion $p^{-1}(B^{n-1}) \to E'$ is a weak homotopy equivalence, which establishes that the rightmost map above is an isomorphism.

   - Use excision and the Kunneth formula to show the leftmost map is an isomorphism (Hint: near each $x_\alpha$, the fibre bundle is just a product).

   - Now let $B$ be a potentially infinite CW complex. Use $n$-connectivity of $(B, B^n)$ to extend the result to a fibration with base $B$.

   - Extend to arbitrary bases $B$ by using CW approximation and considering the pullback bundle.

4. Hatcher 4.D.1, 4.D.2 (applications of the above, including an example where the identification need not be a ring isomorphism).