Math 131A-3: Homework 5

Due: November 1, 2013

1. Do problems 14.2 (a),(f), 14.3(b),(e), 14.4(c), 14.8, 14.12, 14.13 in Ross.

2. Do problems 15.1, 15.4(b) in Ross. [You can use what you know about integration from calculus on 15.4. We’ll define it properly later in this course.]

3. Do problems 17.2, 17.5 in Ross.

4. The number $e$. You have probably seen in calculus that Euler’s number $e$ may be defined as the limit of the sequence $a_n = \left(1 + \frac{1}{n}\right)^n$. This is sometimes described as the interaction between the “irresistible force” – to wit, an exponent approaching infinity – and the “immovable object” – to wit, a base approaching 1. Another possible definition of $e$ is

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

We will show these expressions are both convergent, and in fact coincide. Let $s_n$ be the partial sums of the series $\sum_{n=0}^{\infty} \frac{1}{n!}$.

• (a) Show that $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges. Call the limit $s$.

• (b) The binomial theorem states that, for $n \geq 1$, $(1 + x)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^k$. With this in mind, show that for $n \geq 1$,

$$a_n = \frac{1}{0!} + \sum_{k=1}^{n} \frac{n(n-1)\cdots(n-k+1)}{n^k} \frac{1}{k!}$$

Conclude that $a_n \leq s_n$ for all $n \geq 1$, and therefore $\limsup a_n \leq s$.

• (c) For $n \geq m$, show that

$$a_n \geq \frac{1}{0!} + \sum_{k=1}^{m} \frac{n(n-1)\cdots(n-k+1)}{n^k} \frac{1}{k!}$$

Letting $n \to \infty$ for fixed $m$, observe that we have $\liminf a_n \geq 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{m!}$. Since $m$ was arbitrary, conclude that $\liminf a_n \geq s$.

• (d) From the above, conclude that $(a_n)$ converges and $\lim a_n = s$. Therefore the two definitions of $e$ above are convergent and equal.