Homework 1: Solutions to exercises not appearing in Pressley (also 1.2.3).

Math 120A

- (1.11) (i) Let $\gamma(t) = (1 + \cos t, \sin t(1 + \cos t))$. We set $x = 1 + \cos t$. This implies that $\sin t = \pm \sqrt{2x - x^2}$. Hence we have $y = \pm x\sqrt{2x - x^2}$, or $y^2 = x^2(2x - x^2)$ as a Cartesian equation for the curve.

(ii) Let $\gamma(t) = (t^2 + t^3, t^3 + t^4)$. We first observe that $y = tx$. To exploit this relationship, consider $x^4 = x(x^3) = (t^2 + t^3)(x^3) = t^2x^3 + t^3x^3 = y^2x + y^3$. Ergo the points of our parametrization satisfy $x^4 = y^2x + y^3$. To see that the parametrized curve is the entire set of solutions to this equation, first note that if $x = 0$, then $y = 0$. If $(x, y)$ is a solution to the equation such that $x \neq 0$, write $y = tx$, where $t$ is whatever constant makes this true. Then solving for $x$ shows there is a unique solution to the equation that satisfies $y = tx$ for any particular $t$, and it is $(t^2 + t^3, t^3 + t^4)$.

- (1.12) (i) The tangent vector is $\dot{\gamma}(t) = (-\sin t, \cos t(1 + \cos t) - \sin^2 t) = (-\sin t, \cos t + \cos^2 t - \sin^2 t)$. This vanishes when $t = n\pi$ for $n$ odd.

(ii) The tangent vector is $\dot{\gamma}(t) = (2t + 3t^2, 3t^2 + 4t^3)$, which vanishes when both $t(2 + 3t) = 0 = t^2(3 + 4t)$. This happens only when $t = 0$.

- (1.15) We have $\gamma(t) = (e^t \cos t, e^t \sin t)$, so $\dot{\gamma}(t) = (e^t(\cos t - \sin t), e^t(\sin t + \cos t))$. Their dot product is then $\gamma(t) \cdot \dot{\gamma}(t) = e^{2t}(\cos^2 t - \cos t \sin t + \sin^2 t + \cos t \sin t) = e^{2t}$. Now, $||\gamma(t)|| = e^t$, and $||\dot{\gamma}(t)|| = \sqrt{e^{2t}(\cos^2 t - 2 \sin t \cos t + \sin^2 t + \sin^2 t + 2 \sin t \cos t + \cos^2 t)} = \sqrt{2e^{2t}}$. Hence if the angle between $\gamma(t)$ and $\dot{\gamma}(t)$ is $\theta$, we have $\cos \theta = \frac{\gamma(t) \cdot \dot{\gamma}(t)}{||\gamma(t)|| ||\dot{\gamma}(t)||} = \frac{1}{\sqrt{2}}$. Ergo $\theta = \frac{\pi}{4}$ (where the sign is determined from a look at the graph).

- (1.2.3) Our curve is $\gamma(\theta) = (r(\theta) \cos \theta, r(\theta) \sin \theta)$, for $r$ a smooth function. The tangent vector $\dot{\gamma}(\theta) = (r'(\theta) \cos \theta - r(\theta) \sin \theta, r'(\theta) \sin \theta + r(\theta) \cos \theta)$. For $\gamma$ to be regular, we need both components of the tangent vector to be nonzero everywhere. A computation shows that $||\dot{\gamma}(\theta)||^2 = r(\theta)^2 + r'(\theta)^2$, so the tangent vector can only be zero if $r = r' = 0$ at some $\theta$; as long as this never happens, the curve is regular. For $\gamma$ to be unit speed, we need $r(\theta)^2 + r'(\theta)^2 = 1$. An obvious solution is $r \equiv \pm 1$, which gives the unit circle with some orientation. For the other solution, we separate variables to solve the
differential equation:

\[ 1 = \left( \frac{dr}{d\theta} \right)^2 + r^2 \]
\[ \pm \sqrt{1 - r^2} = \frac{dr}{d\theta} \]
\[ \pm d\theta = \frac{dr}{\sqrt{1 - r^2}} \]
\[ \pm \int d\theta = \int \frac{dr}{\sqrt{1 - r^2}} \]
\[ \pm \theta + C = \sin^{-1}(r) \]
\[ \sin(\pm \theta + C) = r \]

Ergo we have \( \gamma(t) = (\sin(\pm \theta + C) \cos \theta, \cos(\pm \theta + C) \sin \theta) \). This is a circle of radius \( \frac{1}{2} \). For \( \theta + C \), the center is \( \left( \frac{1}{2} \sin C, \frac{1}{2} \cos C \right) \); for \( \text{theta} + C \), the center is \( \left( \frac{1}{2} \sin C, -\frac{1}{2} \cos C \right) \) (This can be checked by showing that the parametrization satisfies the equation of the appropriate circle. A graphical argument was also ok.)

- Question 4. We see that

\[ \frac{\partial f}{\partial x}(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} \]
\[ = \lim_{h \to 0} \frac{0 - 0}{h} \]
\[ = 0 \]

Similarly, \( \frac{\partial f}{\partial y}(0, 0) = 0 \). However, if \( \{(x_n, y_n)\} \) is any sequence of points such that \( x_n = y_n^2 \) and \( y_n \to 0 \), then \( f(x_n, y_n) = \frac{y_n^4}{2y_n^4} = \frac{1}{2} \), so as \( n \to \infty \), \( f(x_n, y_n) \to \frac{1}{2} \neq f(0, 0) \). Ergo \( f \) is not continuous.