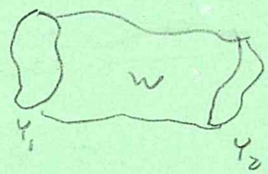
Lecture 9

Last Time

Maps from cobordisms



$$\widehat{F}_W : \widehat{CF}(Y_1) \longrightarrow \widehat{CF}(Y_2)$$

$$F^+_W : CF^+(Y_1) \longrightarrow CF^+(Y_2)$$

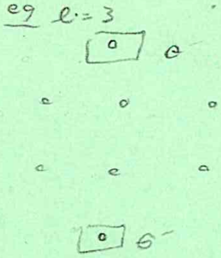
Construction • Decompose W into 1-handles ∪ 2-handles ∪ 3-handles

• One-handles $Y_1 \longrightarrow Y_2 = Y_1 \# (\#_{g-1} (S^1 \times S^2))$

$$\widehat{HF}(Y_1) \longrightarrow \widehat{HF}(Y_2) \simeq \widehat{HF}(Y_1) \otimes (\Lambda^* H^1(\#_{\ell}(S^1\times S^2)))$$

$$\xi \longmapsto \xi \otimes \Theta$$

eg $\ell = 3$



• Two-handles Pick a Heegaard triple diagram rep'ing the cobordism, count triangles.

• Three-handles $Y_1 = Y_2 \# (\#_{g-1} (S^1 \times S^2)) \longrightarrow Y_2$

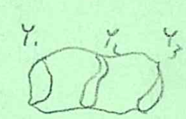
$$\widehat{HF}(Y_2) \otimes (\Lambda^* H^1(\#_{\ell}(S^1\times S^2))) \longrightarrow \widehat{HF}(Y_2)$$

$$\xi \otimes B \longmapsto \xi \otimes \Theta'$$

Today Spin^c-structures & d-invariants.

Claim We can split F_W into $\bigoplus_{\mathfrak{s} \in Spin^c(W)} F_{W,\mathfrak{s}}$ which behave homogeneously

wrt gradings. These are also functorial in the sense that if



$\widehat{F}_{W_1 \# W_2, \mathfrak{s}_1 \# \mathfrak{s}_2} = \widehat{F}_{W_2, \mathfrak{s}_2} \circ \widehat{F}_{W_1, \mathfrak{s}_1}$ and if $b_1(Y_2) = 0$ then $F^+_{(W_1 \# W_2, \mathfrak{s}_1 \# \mathfrak{s}_2)} = F^+_{W_2, \mathfrak{s}_2} \circ F^+_{W_1, \mathfrak{s}_1}$.

What is a Spin^c-structure?

In general, have $1 \to \mathbb{Z}_2 \to \mathrm{Spin}(n) \to SO(n) \to 1$

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}^c(n) \to SO(n) \times U(1) \to 1$$

Special cases

$n=3$)

- $\mathrm{Spin}(3) = SU(2) = S$
- $\mathrm{Spin}^c(3) = U(2)$

$n=4$)

- $\mathrm{Spin}(4) = SU(2) \times SU$
- $\mathrm{Spin}^c(4) = S^1 \times SU(2) \times SU(2)$

Let M closed oriented n-mfld w/ a choice of metric.

$$\begin{array}{ccc} TM & \rightsquigarrow & F_r \\ \downarrow & & \downarrow \\ M & & M \end{array}$$ principal $SO(n)$-bundle.

- A spin structure is a lift of F_r to a principal $\mathrm{Spin}(n)$ bundle (a double cover)
- A spin^c-structure is a lift of F_r to a principal $\mathrm{Spin}^c(n)$-bundle
- So a $\mathrm{Spin}^c(n)$-bundle $\begin{array}{c} F \\ \downarrow \\ M \end{array}$ w/ an isomorphism $\begin{array}{c} F/U(1) \\ \downarrow \\ M \end{array} \simeq F_r$

(That is, the map $\begin{array}{c} F_r \\ \downarrow \\ F \end{array}$ is $\begin{array}{c} \mathrm{Spin}^c(4) \\ \downarrow \\ SO(4) \end{array}$ on each fibre.)

$$\mathrm{Spin}^c(4) = \{ (A,B) \in U(2) \times U(2) : \det(A) = \det(B) \}$$

We have a homomorphism $\alpha : \mathrm{Spin}^c(4) \to S^1$ and we can
$$(A,B) \mapsto \det A$$

Use this to get a line bundle $\left.\begin{array}{c} L = F \times_\alpha \mathbb{C} \\ \downarrow \\ X \end{array}\right\}$ determinant line bundle

Then we have $c_1(L) = c_1(s)$

There is an exact sequence

$$1 \longrightarrow \mathbb{Z}_2 \to Spin^c(4) \longrightarrow S^1 \times SO(4) \longrightarrow 1$$

The principal bundle P of $L \times TM$ is an $\underset{SO(2)}{\underset{\parallel}{S^1}} \times SO(4)$ principal bundle. This

double cover $Spin^c(4) \longrightarrow SO(4)$ is the unique double cover that

extends over $SO(6)$.

This implies that Fr has a spin^c-structure w/ determinant line bundle L $\Leftrightarrow$ P has a spin structure $\Leftrightarrow$ $w_2(P) = 0$ $\Leftrightarrow$ $w_2(L) + w_2(TM) = 0$ $\Rightarrow$ $w_2(L) \equiv w_2(TM) \pmod 2$

$\Leftrightarrow$ $w_2(TM) = c_1(L) \bmod 2$.

This is exactly the set of characteristic vectors for the intersection form.

A spin^c-structure on M restricts to a spin^c-structure on the bdy by restricting.

Decomposing the cobordism map:

- In the one or three-handle case:

$$Y_1 \# S^3 \downarrow Y_1 \# (\#_\ell S^1 \times S^2)$$

$W = (Y_1 \times I) \# (\#_\ell D^1 \times D^3)$

Exactly one spin^c structure on Y_1 extends over $W \rightsquigarrow$ there is a canonical s in which $F_{W,s} \neq 0$.

- Two handles: consider the Heegaard triple diagram $(\Sigma_g, \alpha, \beta, \gamma)$,

Recall
- This specifies a 4-mfd $X_{\alpha\beta\gamma}$ w/ $\partial = -Y_{\alpha\beta} - Y_{\beta\gamma} + Y_{\alpha\gamma}$
- Fill in $\#_{g-\ell} (S^1 \times D^3)$ to obtain $W \rightsquigarrow \mathrm{Spin}^c(W) \simeq \mathrm{Spin}^c(X_{\alpha\beta\gamma})$
- Consider the group of periodic domains $P = \{ \phi : \partial D(\phi)$ is a linear combination of curves $\alpha_i, \beta_i, \gamma_i \}$
- Exercise: $P \simeq H_2(X_{\alpha\beta\gamma})$
- Exercise: If we quotient by the set Q of disks that are periodic domains in one of the 3-mfds, then $P/Q \simeq H^2(X_{\alpha\beta\gamma})$.

Let $x, x' \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $y, y' \in \mathbb{T}_\beta \cap \mathbb{T}_\gamma$, $z, z' \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$ be in the same spin^c-structures in each case.

Then given $\psi \in \pi_2(x, y, z)$, $\psi' \in \pi_2(x', y', z')$, there is a difference $\delta(\psi, \psi') \in H^2(X_{\alpha\beta\gamma})$ corresponding to $D(\psi) - D(\psi') + D(\phi_1) + D(\phi_2) + D(\phi_3)$.

Similar to the 3-mfd case but somewhat more involved:

$\exists\ \psi \in \pi_2(x, y, z) \iff \exists\ s \in Spin^c(W)$ st $s|_{Y_{\alpha\beta}} = s_z(x)$, $s|_{Y_{\beta\gamma}} = s_z(y)$, $s|_{Y_{\alpha\gamma}} = s_z(w)$

Intersection Forms

Recall Let X be a cpt oriented 4-mfd. There is an intersection form $Q_X : H^2(X, \partial X; \mathbb{Z})/_{Tors} \times H^2(X, \partial X; \mathbb{Z})/_{Tors} \to \mathbb{Z}$

$$\xi \otimes \eta \longmapsto \langle \xi \cup \eta, [X] \rangle = \xi \cdot \eta$$

Equivalently $H_2(X; \mathbb{Z})/_{Tors} \otimes H_2(X; \mathbb{Z}) \to \mathbb{Z}$ (mod tors)

If ∂X is a $\mathbb{Q}HS^3$, this becomes a pairing

$$Q_M : H^2(X; \mathbb{Q}) \times H^2(X; \mathbb{Q}) \longrightarrow \mathbb{Q}$$

$$\overset{\cong}{H^2(X, \partial X; \mathbb{Q})}$$

A characteristic vector for this form is one for which $\xi \cdot v = v \cdot v$ (mod 2) for every $H_2(X; \mathbb{Z})$ (mod tors)

- More generally:
- $\sigma(M)$ is the signature of this intersection form.
- $\xi^2 = Q_M(\xi, \xi) \in \mathbb{Q}$. If $n\xi|_{\partial M} = 0$, then $n\xi^2 \in \mathbb{Z}$.

Propn If Y is a rational homology 3-sphere, $\exists!$ $\mathbb{Q}$ lift of the relative $\mathbb{Z}$-grading on $HF^+(Y, t)$ w/ the following properties

- $\widehat{HF}(S^3) \cong \mathbb{F}_{(0)}$
- $\widehat{CF}(Y, t) \longrightarrow CF^+(Y, t)$ is degree-preserving
- If ξ is homogeneous in $CF^+(Y, t)$, then

$$gr\, f^+_{W,s}(\xi) - gr(\xi) = \frac{c_1(s)^2 - 2\chi(W) - 3\sigma(W)}{4}$$

Defn Pick a cobordism $W : S^3 \to Y$ consisting of 2-handles and find corresponding Heegaard triple $(\Sigma, \alpha, \beta, \gamma)$, so that $Y_{\alpha\beta} = S^3$ and $Y_{\beta\gamma} = \#_{\ell} (S^1 \times S^2)$. Choose $\psi \in \pi_2(\hat{\theta}_{\alpha\beta}, \hat{\theta}_{\beta\gamma}, x)$. Then

$$gr(\vec{x}) = -\mu(\psi) + 2n_z(\psi) + \frac{c_1(s)^2 - 2\chi(W) - 3\sigma(W)}{4}.$$

One then checks (somewhat lengthily) that this is an invariant.

The d-invariant

Propn If Y is a rational homology sphere, then the structure of $HF^+(Y,s)$ is

$$\mathbb{F}[U,U^{-1}]_{(d)} / U\mathbb{F}[U] \oplus \Big(\underbrace{\bigoplus_{i=1}^{m} \mathbb{F}[U]_{(s_i)} / U^{n_i}}_{\text{noncanonically}} \Big)$$

Or if you prefer, $U^n HF^+(Y,s) \simeq \mathbb{F}[U,U^{-1}] / U\mathbb{F}[U]$ for $n \gg 0$.

- We say $d(Y,s)$ is the minimum grading in $U^n HF^+(Y)$.
- Note that $d(Y_1 \# Y_2, s_1 \# s_2) = d(Y_1, s_1) + d(Y_2, s_2)$.

Equivalently $d(Y,t)$ is the minimal k st $\pi_k : HF^\infty_k(Y,t) \to HF^+_k(Y,t)$ has nontrivial image.

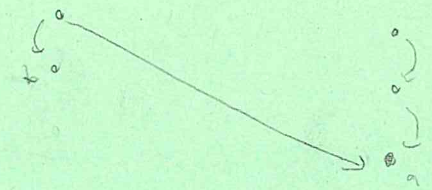
Or 2 more than the maximal ℓ st $\iota_\ell : HF^-_\ell(Y,t) \to HF^\infty_\ell(Y,t)$ has nontrivial image.

Thm If s is torsion, then $HF^\infty(Y,s) \simeq \mathbb{F}[U,U^{-1}]$.

Thm If W is a cobordism from (Y_1, t_1) to (Y_2, t_2) w/ $b_2^+(W) = 0$, then the map $F^\infty_{W,s} : HF^\infty(Y_1, t_1) \longrightarrow HF^\infty(Y_2, t_2)$ is an isomorphism for any s st $s|_{Y_i} = t_i$.

Corollary If (Y_1, t_1) and (Y_2, t_2) are connected by a rational homology cobordism, then $d(Y_1, t_1) = d(Y_2, t_2)$. In particular if Y bounds a rational homology ball, then $d(Y, t) = 0$.

$$\begin{tikzcd} HF^\infty_i(Y_1, t_1) \arrow[r, "\sim"] \arrow[d] & HF^\infty_i(Y_2, t_2) \arrow[d] \\ HF^+_i(Y_1, t_1) \arrow[r] & HF^+(Y_2, t_2) \end{tikzcd}$$



$$\Rightarrow d(Y_2, t_2) - d(Y_1, t_1) \geq \frac{\overset{0}{\cancel{c_1(s)^2}} - 2\chi(W) - 3\sigma(W)}{4} \Big\} -2(b_2^+ + b_2^-) - 3(b_2^+ - b_2^-) = b_2^- = b_2$$

$\Rightarrow d(Y_2, t_2) \geq d(Y_1, t_1)$.

But we could reverse the orientation of $W \Rightarrow d(Y_2, t_2) = d(Y_1, t_1)$.

Corollary Suppose Y is an integer homology 3-sphere, then for each negative-definite 4-mfd X bounding Y, we have

$$Q_X(\xi, \xi) + \underbrace{H^2(X; \mathbb{Z})}_{b_2} \leq 4d(Y)$$

[So if $d(Y) < 0$, there is no negative-definite X w/ $\partial X = Y$.]