Lecture 8

Recall: Last time (Y, Y_0, Y_1) a triad given by filling M (w/ $\partial M = S^1 \times S^1$) along $\gamma, \gamma_0, \gamma_1$ w/ $\gamma \cdot \gamma_0 = \gamma_0 \cdot \gamma_1 = \gamma_1 \cdot \gamma = -1$.

A triple $(\Sigma, \alpha, \beta, \gamma, z)$ specifies a 4-mfd X w/ bdy $-Y_{\alpha\beta} - Y_{\beta\gamma} + Y_{\alpha\gamma}$.

Counting triangles we get a map

$$\widehat{CF}(Y_{\alpha\beta}) \otimes \underset{\substack{\uparrow \\ \text{standard input here}}}{\widehat{CF}(Y_{\beta\gamma})} \to \widehat{CF}(Y_{\alpha\gamma})$$

$$\rightsquigarrow F : \widehat{HF}(Y_{\alpha\beta}) \to \widehat{HF}(Y_{\alpha\gamma})$$

We have a triangle

$$\begin{tikzcd} \widehat{HF}(Y_{\alpha\beta}) \arrow[rr, "F"] & & \widehat{HF}(Y_{\alpha\gamma}) \arrow[dl, "F_0"] \\ & \widehat{HF}(Y_{\alpha\delta}) \arrow[ul, "F_1"] & \end{tikzcd}$$

Last time: This is a chain cpx.

Last time: Computed for $S^3_{-1}(U)$ by hand. Exercise: Compute for $S^3_p(U)$.

Today: Why is this sequence exact? (Sketch)

Idea: Prove that $M(F)$ is chain homotopy equivalent to $\widehat{CF}(Y_{\alpha\delta})$.

Why? We always have an ses, for $F: A \to B$ a chain map

$$0 \to B \to M(F) \to A \to 0$$

and a corresponding les.

Homological algebra

Suppose that $\{A_i\}_{i=1}^{\infty}$ be a collection of chain maps and let $\{F_i : A_i \longrightarrow A_{i+1}\}_{i \in \mathbb{Z}}$ be a collection of chain maps st:

① $F_{i+1} \circ F_i$ is chain htp'c to zero via some chain homotopy

$$H_i : A_i \longrightarrow A_{i+2}.$$

② $\Psi_i : F_{i+2} \circ H_i + H_{i+1} \circ F_i : A_i \longrightarrow A_{i+3}$ is a quasi-isomorphism.

Then $H_*(M(F_i)) \cong H_*(A_{i+2})$

<u>Exercise</u> Ψ_i is a chain map, and (two applications of) the Five lemma shows the statement is true.

Finishing the surgery triangle proof

- To do this correctly, we pick infinitely many small translates of β, γ, δ, so that all the A_i will come from distinct curves

$$A_{3i+1} = \widehat{CF}(Y_{\alpha\gamma^{(i)}}) = \widehat{CF}(Y_0)$$

$$A_{3i+2} = \widehat{CF}(Y_{\alpha\delta^{(i)}}) = \widehat{CF}(Y_1)$$

$$A_{3i+3} = \widehat{CF}(Y_{\alpha\beta^{(i)}}) = \widehat{CF}(Y_\infty)$$

We already checked ①; $H_i : A_{3i} \longrightarrow A_{3i+2}$ is $h_{\alpha,\beta^{(i)},\gamma^{(i)},\delta^{(i)}}(\xi \otimes \Theta_{\beta^{(i)}\gamma^{(i)}} \otimes \Theta_{\gamma \delta}$

For the second hypothesis, we use the <u>area filtration</u>.

<u>Recall</u> An $\mathbb{R}$-Filtration of a group G is a sequence of subgroups indexed by $r \in \mathbb{R}$ st

- $G_r \subseteq G_s$ if $r \leq s$
- $G = \bigcup_{r \in \mathbb{R}} G_r$

This induces a partial ordering on G: $x < y$ if $\exists\, r$ st $x \in G_r$ but $y \notin G_r$.

We have a filtration of $\widehat{CF}(Y_{\alpha\beta})$ via fixing an intersection points $\vec{x}_0$ and letting $\mathcal{F}: \mathbb{T}_\alpha \cap \mathbb{T}_\beta \longrightarrow \mathbb{R}$ by $\vec{x}$ goes to $\vec{\mathcal{F}}(\vec{x}) = A(D(\phi)) - 2n_z(\phi) \cdot A(\Sigma)$ for some $\phi \in \pi_2(x, x_0)$.

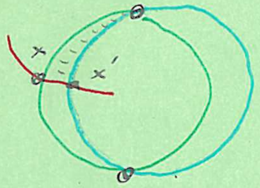
[If $b_1 = 0$, we need to pick a form st $A(P) = 0$ for any $P \in \pi_2(x, x)$]

<u>Lemma</u> If β' is a small perturbation of β, then the chain map

$$\widehat{CF}(Y_{\alpha\beta}) \longrightarrow \widehat{CF}(Y_{\alpha\beta'})$$
$$\xi \longmapsto F_{\alpha\beta\beta'}(\xi \otimes \Theta_{\beta\beta'})$$

is an isomorphism on homology.

<u>Pf</u>



There is a unique closest point map ι.

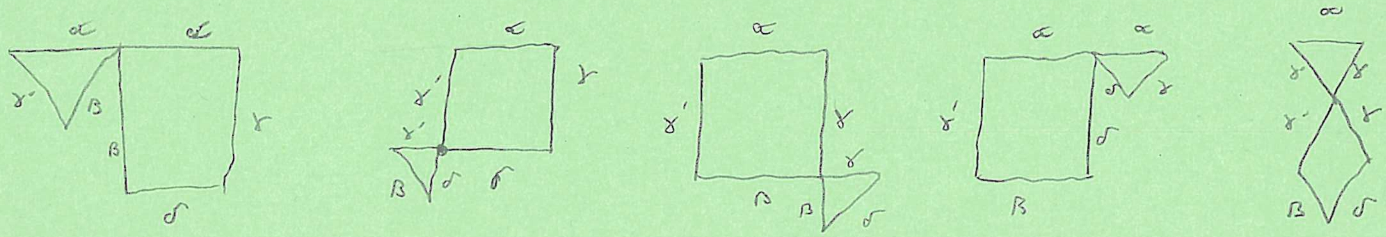
We can compare to $F_{\alpha\beta\beta'}(\xi \otimes \Theta_{\beta\beta'}) = \iota(\xi)$.

The difference $F - \iota$ is some sum y of elements st $\mathcal{F}(\iota(x)) \leq \mathcal{F}(y)$ in the area filtration.

$\rightsquigarrow$ since ι is an isomorphism on the group level, so is F.

This gives us maps $\theta_i : A_i \longrightarrow A_{i+3}$ via this isomorphism.

The claim here is that $F_3 \circ H_1 + H_2 \circ F_1 : A_1 \longrightarrow A_4$ is chain homotopy equivalent via counting pseudohol'c maps of <u>pentagons</u> between $(\alpha, \gamma, \delta, \beta, \gamma')$.



$\rightsquigarrow$ Also any copy of 

<u>We get a nulhomotopy of F</u>

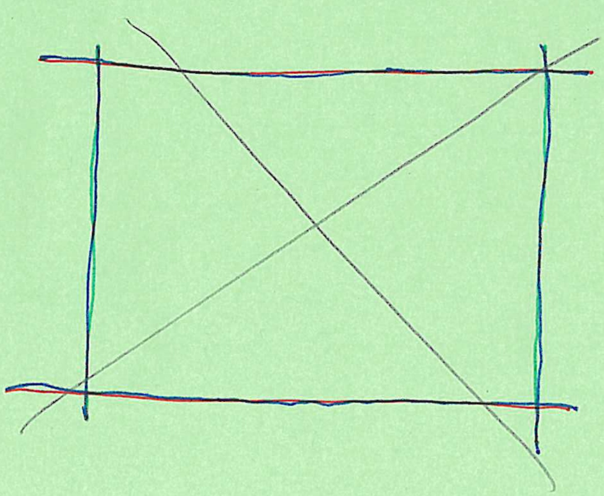
$F_{\alpha\beta\gamma^{(1)}}\left(h_{\alpha\gamma\delta\beta}(\xi \otimes \theta_{\gamma\delta} \otimes \theta_{\delta\beta}) \otimes \theta_{\beta\gamma^{(1)}}\right)$ $\}$ $F_3 \circ H_1$

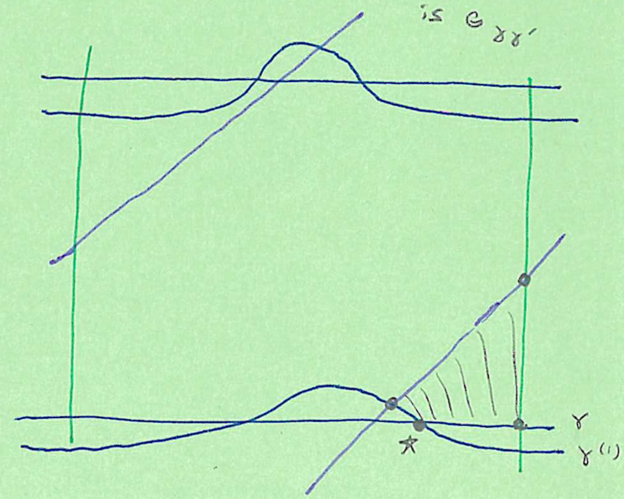
$+\, h_{\alpha\gamma\delta\gamma^{(1)}}\left(\xi \otimes \theta_{\gamma\delta} \otimes F_{\delta\beta\gamma^{(1)}}(\cancelto{0}{\theta_{\delta\beta} \otimes \theta_{\beta\gamma^{(1)}}})\right)$

$+\, h_{\alpha\gamma\beta\gamma^{(1)}}\left(\xi \otimes \hat{F}_{\gamma\delta\beta}(\cancelto{0}{\theta_{\gamma\delta}} \otimes \theta_{\delta\beta}) \otimes \theta_{\beta\gamma}\right)$

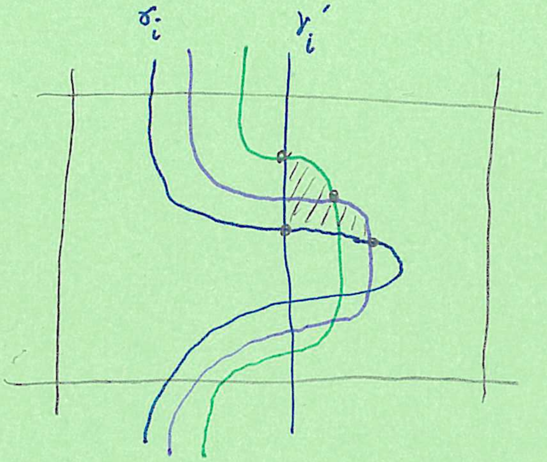
$+\, h_{\alpha\delta\beta\gamma^{(1)}}\left(F_{\alpha\gamma\delta}(\xi \otimes \theta_{\gamma\delta}) \otimes \theta_{\delta\beta} \otimes \theta_{\beta\gamma}\right)$ $\}$ $H_2 \circ F_1$

$+\, F_{\alpha\gamma\gamma^{(1)}}\left(\xi \otimes h_{\gamma\delta\beta\gamma^{(1)}}(\theta_{\gamma\delta} \otimes \theta_{\delta\beta} \otimes \theta_{\beta\gamma^{(1)}})\right)$ $\leftarrow$ <u>Claim</u> This is θ_1, i.e. the inner term is $\theta_{\gamma\gamma'}$





In the general case this also counts



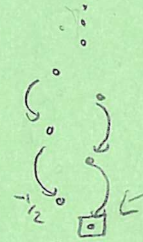
$\leadsto$ now this follows from homological algebra.

In the $HF^{\pm}$ case, one has to be more careful that all sums involved in the maps f are actually finite.

<u>Example</u>

- The standard proof of $HF^+(\Sigma(2,3,5))$ involves showing that for a knot w/ a lens space surgery, to $L(p,q)$

$HF^+(Y_0(K), [0]) =$



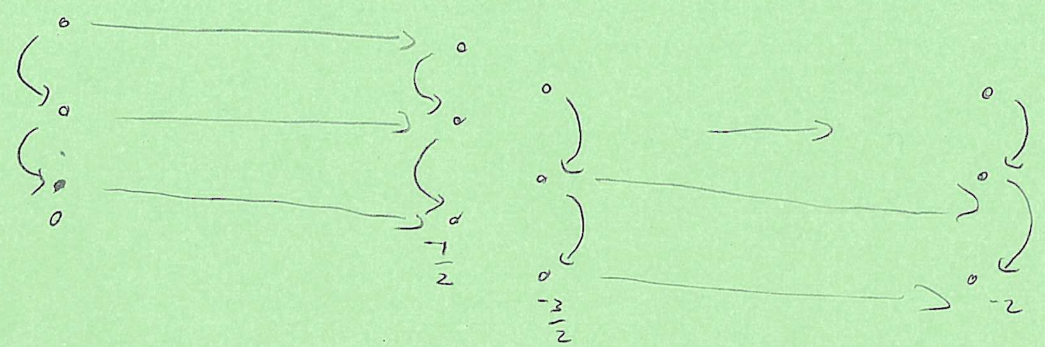
$d(L(p,q),[0]) - d(L(p,1),[0]) + \frac{1}{2}$

$\leadsto \overset{RHT}{3_1}$ has a lens space surgery ($S^3_{+5}(3_1)$) to $L(\, , \,)$

$\leadsto HF^+(Y_0(3_1), [0])$

grading drops by 1/2 grading drops by $\frac{1}{2}$

$$\cdots \longrightarrow HF^+(S^3) \longrightarrow HF^+(S^3_0(RHT), [0]) \longrightarrow HF^+(S^3_{+1}(RHT)) \longrightarrow$$

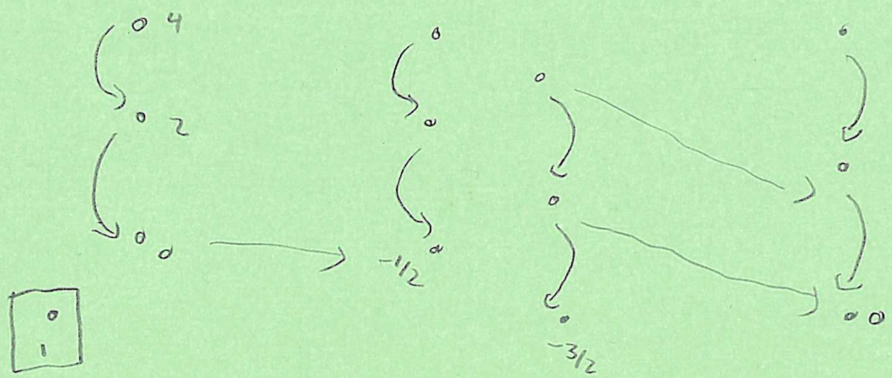


$(S^3, S^3_0(RHT), S^3_{+1}(RHT))$

<u>Example</u> $\Sigma(2,3,7)$

grading drops by $\frac{1}{2}$ grading drops by $\frac{1}{2}$

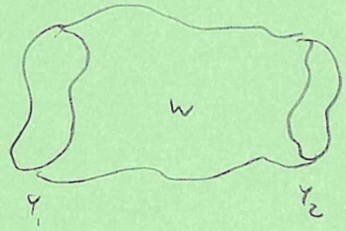
$$\longrightarrow HF^+(\Sigma(2,3,7)) \longrightarrow HF^+(Y_0) \longrightarrow HF^+(S^3) \longrightarrow$$



$(S^3, S^3_{-1}(RHT), S^3_0(RHT))$

Moie generally

There is a map $\hat{F}_{W,s} : \widehat{HF}(Y_1, s|_{Y_1}) \longrightarrow \widehat{HF}(Y_2, s|_{Y_2})$ For any cobordism W



IF W is between rational homology spheres,

$$gr(\hat{F}_{W,s}(\xi) - gr(\xi)) = \frac{c_1(s)^2 - 2\chi(W) - 3\sigma(W)}{4}$$

For $W = S^3 \cup$ 2-handle $= D^2 \times S^2 - \{ball\}$

$$\frac{c_1(s)^2 - 2\chi(W) - 2\sigma(W)}{4} = \frac{0-2-0}{4} = \frac{-1}{2}$$

- For a set of 1-handles: $Y_2 \simeq Y_1 \#_{\ell} (S^1 \times S^2)$

$$\rightsquigarrow \widehat{HF}(Y_2) \cong \widehat{HF}(Y_1) \otimes \Lambda^* H^1(\#^1 (S^1 \times S^2))$$

Θ a top-degree generator

$$\hat{F}_{W,s}(\xi) = \xi \otimes \Theta$$

- For a set of 3-handles: $Y_1 \simeq Y_2 \# S^1 \times S^2$. $\hat{F}_W : \widehat{HF}(Y_1) \longrightarrow \widehat{HF}(Y_2)$ via projection onto the bottom-degree element in $\Lambda^* H^1(\# (S^1 \times S^2))$

- For a set of 2-handles: We Find a Heegaard triple diagram that includes each of the attaching circles