Recall last time \((Y, y_0, y_1)\) a triad given by filling \(M = (u/M, s_0, s_1)\) along \(\delta, \gamma, \delta\) with \(\delta \gamma_0 = \delta \gamma_1 = \delta \gamma = 1\).

A triple \((\xi, \alpha, \beta, \delta, \gamma)\) specifies a 4-mfbd\(\breve{X}\) w/ body \(-Y_{ab} - Y_{ba} + Y_{ab}\).

Counting triangles we get a map
\[
\mathcal{C}(\hat{Y}_{ab}) \otimes \mathcal{C}(\hat{y}_{ab}) \rightarrow \mathcal{C}(\hat{Y}_{ab})
\]
standard input here

\(\text{Last time } F : \hat{H}_F (Y_{ab}) \rightarrow \hat{H}_F (Y_{ab})\)

\(\text{We have a triangle}\)

\[
\begin{array}{ccc}
\hat{H}_F (Y_{ab}) & \xrightarrow{F} & \hat{H}_F (y_{ab}) \\
\hat{H}_F (\bar{Y}_{ab}) & \xleftarrow{F_i} & \hat{H}_F (\bar{Y}_{ab}) \\
\hat{H}_F (Y_{ab}) & \xleftarrow{F_0} & \hat{H}_F (Y_{ab})
\end{array}
\]

\(\text{Last time } \text{Computed For } S^3_\epsilon (U) \text{ by hand. Exercise: Compute For } S^3_p (U)\).

Today: Why is this sequence exact? (Sketch)

Idea: Prove that \(M(F)\) is chain homotopy equivalent to \(\mathcal{C}(\hat{Y}_{ab})\).

Why? We always have an SES, For \(F : A \rightarrow B\) a chain map
\[
0 \rightarrow B \rightarrow M(F) \rightarrow A \rightarrow 0
\]
and a corresponding yes.
Homological algebra

Suppose that $\xi: \{ F_i \}$ be a collection of chain maps and let $\xi: F_i: A_i \to A_{i+1}$ be a collection of chain maps on $A_i$.

1. $F_i$ of $\xi$ is chain homotopic to zero via some chain homotopy $H_i: A_i \to A_{i+1}$.

2. $Y_i: F_{i+2} \circ H_{i+1} \circ F_i: A_i \to A_{i+3}$ is a quasi-isomorphism.

Then $H_0(A_0(Y_i)) \cong H_0(A_{i+2})$.

Exercise $Y_i$ is a chain map, and (two applications of) the Five Lemma shows the statement is true.

Finishing the surgery triangle proof

To do this correctly, we pick infinitely many small translates of $\beta, \gamma, \delta$ so that all the $A_i$ will come from distinct curves $\hat{\alpha}$ and $\hat{\beta}$.

$A_3(x, y) = \hat{\alpha}(x, y) = \hat{\beta}(y)$

For the second hypothesis, we use the area filtration.
Recall an $\mathcal{F}$-filtration of a group $G$ is a sequence of subgroups indexed by $r \in \mathbb{R}$ such that:

- $G_r \leq G_s$ if $r < s$
- $G = \bigcup_{r \in \mathbb{R}} G_r$

This induces a partial ordering on $G$: $x \leq y$ if $r \geq s$ for $x \in G_r$, but $y \in G_s$.

We have a filtration on $\hat{CF}(Y_{ab})$ via fixing an intersection point $x_0$ and letting $F: \pi_1 \to \mathbb{R}$ by $x$ goes to $\hat{F}(x) = A(D(x)) = 2n_2(x), A(x)$ for some $\theta \in \pi_2(x, x_0)$.

Proof: If $b = 0$, we need to pick a form $A$ such that $A(p) = 0$ for any $p \in \pi_2(x, x_0)$.

Lemma. If $B'$ is a small perturbation of $B$, then the chain map

\[ \hat{CF}(Y_{ac}) \to \hat{CF}(Y_{ac'},) \]

\[ x \to \hat{F}_{B \to B'}(\xi \otimes \theta_{BB'}) \]

is an isomorphism on homology.

Proof.

There is a unique closest point map $F$.

We can compare to $\hat{F}_{B \to B'}(\xi \otimes \theta_{BB'}) = \delta(\xi)$. The difference $F - \delta$ is some sum of elements $y \in F(c(x)) \leq F(y)$ in the area filtration.

Since $\delta$ is an isomorphism on the group level, so is $F$. 
This gives us maps $\theta_i : A_i \rightarrow A_{i+2}$ via this isomorphism.

The claim here is that $F_3 \circ H_1, H_2 \circ F_1 : A_i \rightarrow A_{i+2}$ is chain homotopy equivalent via counting pseudoholomorphic maps of pentagons between $(x, y, d, \beta, y')$.

$\triangleright$ Also any copy of

We get a nullhomotopy of

$F_{\alpha \beta \gamma} \circ (\hat{h}_{\alpha \beta} \circ (\hat{e}_x \circ \hat{e}_y \circ \hat{e}_z \circ \hat{e}_w \circ \hat{e}_v) \circ \theta_{\gamma \beta \alpha}) = \int F_3 \circ H_1$

$+ \hat{h}_{\alpha \beta \gamma} \circ (\hat{e}_x \circ \hat{e}_y \circ \hat{e}_z \circ \hat{e}_w \circ \hat{e}_v) \circ \theta_{\gamma \beta \alpha}$

$+ \hat{h}_{\alpha \beta \gamma} \circ F_{\alpha \beta} \circ (\hat{e}_x \circ \hat{e}_y \circ \hat{e}_z \circ \hat{e}_w \circ \hat{e}_v) \circ \theta_{\gamma \beta \alpha}$

$+ \hat{h}_{\alpha \beta \gamma} \circ F_{\alpha \beta} \circ (\hat{e}_x \circ \hat{e}_y \circ \hat{e}_z \circ \hat{e}_w \circ \hat{e}_v) \circ \theta_{\gamma \beta \alpha}$

$= \int H_2 \circ F_1$

$+ F_{\alpha \beta \gamma} \circ (\hat{e}_x \circ \hat{e}_y \circ \hat{e}_z \circ \hat{e}_w \circ \hat{e}_v) \circ \theta_{\gamma \beta \alpha}$

$\triangleright$ Claim. This is $0$, i.e.,

The inner term is $0$. \hspace{1cm}$\triangleright$

$\triangleright$ \hspace{1cm}$\triangleright$
In the general case, this also counts

We now this follows from homological algebra.

In the $\mathbb{HF}^+$ case, one has to be more careful that all sums involved in the maps $\mathcal{F}$ are actually finite.

**Example**

The standard proof of $\mathbb{HF}^+(\Sigma(2,3,5))$ involves showing that for a knot with a lens space surgery, to $L(p,q)$

$$\mathbb{HF}^+(\Sigma(p,q), \mathcal{D}) = \{ \text{link} \}$$

In $\mathbb{HF}^+$, has a lens space surgery $(S^3, (3,1))$ to $L(\cdot, \cdot)$

$$m) \mathbb{HF}^+(\Sigma(3,1), \mathcal{D})$$

\[ \ldots \text{diagram} \ldots \]
\[ \cdots \to HF^+(S^3) \to HF^+(S^3_{o} (RHT), \mathbb{I}) \to HF^+(S^3_{+1} (RHT)) \to \]

\[ (S^3, S^3_{o} (RHT), S^3_{+1} (RHT)) \]

Example \[ \Sigma (2, 3, 7) \]

\[ \to HF^+(\Sigma (2, 3, 7)) \to HF^+(\Sigma_0) \to HF^+(S^3) \to \]

\[ (S^3, S^3_{-1} (RHT), S^3_{o} (RHT)) \]
More generally

There is a map \( \hat{F}_{w,s} : \hat{HF}(Y, sl_w) \to \hat{HF}(Y_2, sl_{Y_2}) \) for any cobordism \( W \)

If \( W \) is between rational homology spheres,

\[
g_t(\hat{F}_{w,s}(Y) - \hat{F}(Y)) = \frac{c_t(s)^2 - 2 \chi(W) - 3 \sigma(W)}{4}
\]

For \( W = S^3 \cup 2\text{-handle} = D^2 \times S^2 = \# \text{ball} \)

\[
\frac{c_t(s)^2 - 2 \chi(W) - 2 \sigma(W)}{4} = \frac{0-2-0}{4} = -\frac{1}{2}
\]

For a set \( 1\text{-handles} : Y_2 \cong Y \# (S^1 \times S^2) \)

\[
\hat{H}_1(Y_2) \cong \hat{H}_1(Y) \otimes \Lambda^* H^* (\# (S^1 \times S^2))
\]

\( \otimes \) a top-degree generator

\[
\hat{F}_{w,s}(Y_2) = \otimes \otimes
\]

For a set \( 2\text{-handles} : Y_1 \cong Y_2 \# S^1 \times S^2 \), \( \hat{F}_w : \hat{HF}(Y_1) \to \hat{HF}(Y_2) \) via projection onto the bottom-degree element in \( \Lambda^* H^* (\# (S^1 \times S^2)) \)

For a \( 2\text{-handles} \): We find a Heegaard triple diagram that includes each of the attaching circles.