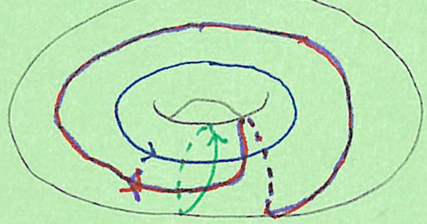
Lecture 7

This week: the surgery exact triangle and maps associated to cobordisms.

$K \hookrightarrow Y^3$ Consider 3 curves on $\partial(Y - \nu K)$: • μ a meridian (bounds a disk in Y but not $Y \setminus \nu K$)

• λ a choice of longitude, oriented so that $\mu \cap \lambda = -1$

• $(\lambda + \mu)$



Let Y_∞, Y_0, Y_1 be Dehn fillings of $Y - \text{nbd}(K)$ w/ slopes $1\mu + 0\lambda$, $1\lambda + 0\mu$, $\lambda + 1\mu = \lambda + \mu$.

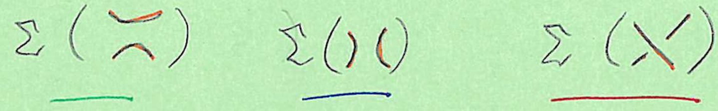
This is a triad. In general (Y, Y_0, Y_1) is a triad if there is a 3-mfd M w/ torus bdy st there are 3 simple closed curves $\gamma, \gamma_0, \gamma_1$ w/ $\gamma \cdot \gamma_0 = \gamma_0 \cdot \gamma_1 = \gamma_1 \cdot \gamma = -1$ and st Y, Y_0, Y_1 are obtained from M by attaching a torus along $\gamma, \gamma_0, \gamma_1$.

Examples of triads • S^3, $S^3_p(K)$, $S^3_{p+1}(K)$ for $p \in \mathbb{Z}$

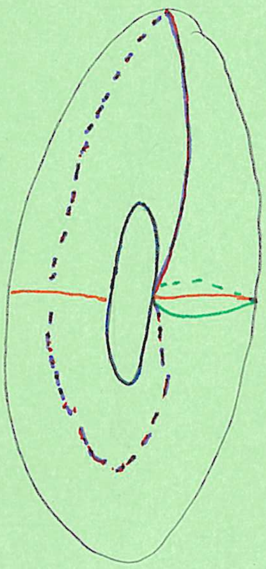
• More generally $S^3_{p_1/q_1}(K)$, $S^3_{p_2/q_2}(K)$, $S^3_{\frac{p_1+p_2}{q_1+q_2}}(K)$

where $p_1 q_2 - q_1 p_2 = 1$.

Importantly: For $L \subseteq S^3$



If we cut out  from $\Sigma(L)$, we cut out a torus from $\Sigma_r(L)$.



Exercise If (Y, Y_0, Y_1) is a triad, then

$$|H_1(Y)| = |H_1(Y_0)| + |H_1(Y_1)|$$

after possible re-ordering.

Exercise If (Y, Y_0, Y_1) are a triad ordered so the equation above is true, and Y_0, Y_1 are L-spaces, then so is Y.

$\rightsquigarrow$ if $S_r(K)$ is an L-space, and $s > r$, then so is $S_s(K)$. Example $S^3_{pq-1}(T_{p,q})$ is a lens space, so for $s > pq-1$, $S^3_r(T_{p,q})$ is an L-space.

Thm There is an exact sequence

$$\begin{tikzcd} \widehat{HF}(Y) \arrow[rr] & & \widehat{HF}(Y_0) \arrow[dl] \\ & \widehat{HF}(Y_1) \arrow[ul] & \end{tikzcd}$$

Strategy Find a Heegaard diagram in which $\beta_1 = \mu$. Then modify to get diagrams for Y_0 and Y_1.

Let γ_i = pushoff of β_i for $i > 1$

Let $\gamma_1 = \lambda$

Let δ_i = pushoff of β_i for $i > 1$

Let $\delta_1 = \lambda + \mu$.

So $Y_{\alpha\beta} = Y_0 \quad Y_{\alpha\delta} = Y_1 \quad Y_{\alpha\beta} = Y$

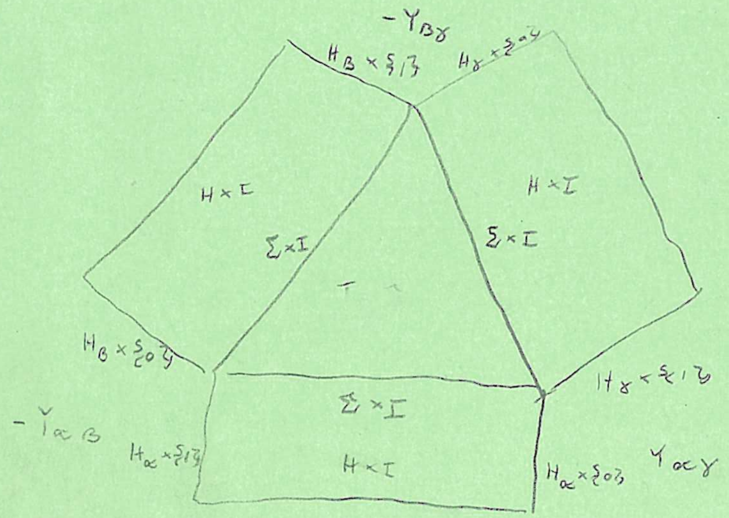
α's not shown

We say $(\Sigma, \alpha, \beta, \gamma)$ is a Heegaard triple diagram if each of the sets $\{\alpha\}, \{\beta\}, \{\gamma\}$ is linearly independent in $H_1(\Sigma)$ and disjoint

Such a diagram specifies a four-manifold X, with

$\partial X = -Y_{\alpha\beta} - Y_{\beta\gamma} + Y_{\alpha\gamma}$: Start w/ $\Delta \times \Sigma = T$, w/ bdy

$$\begin{cases} e_\alpha \times \Sigma \\ e_\beta \times \Sigma \\ e_\gamma \times \Sigma \end{cases}$$



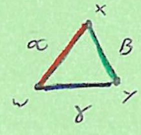
Glue on three copies of $H \times I$, where H is a genus g handlebody.

(Misleading: the outer edge doesn't actually exist.)

Use this to define a map

$$F_{\alpha\beta\gamma} : CF^\circ(\alpha, \beta) \otimes CF^\circ(\beta, \gamma) \longrightarrow CF^\circ(\alpha, \gamma)$$

• In $Sym^g(\Sigma)$, we have $\mathbb{T}_\alpha$, $\mathbb{T}_\beta$, $\mathbb{T}_\gamma$. If $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $y \in \mathbb{T}_\beta \cap \mathbb{T}_\gamma$, $w \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$, then we consider π_2

$\pi_2(x, y, w) = \{$ homotopy classes of maps ψ  $\to Sym^g(\Sigma)\}$

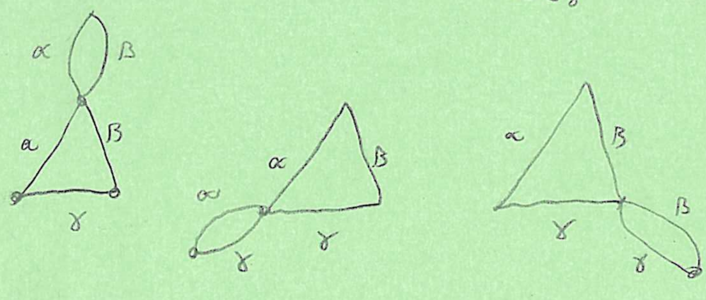
Notice that this does not come w/ a need to reparametrize; we get 0-dim'l compact moduli spaces of hol'c reps of such maps.

For the hat version:

Let $f_{\alpha,\beta,\gamma}(x,y) := \sum_{w \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\psi \in \pi_2(x,y,w) \\ n_z(\psi)=0 \\ \mu(\psi)=0}} \#\mathcal{M}(\psi) \cdot w$, extend bilinearly.

Claim This commutes w/ $\partial_{\alpha\beta} \otimes \partial_{\beta\gamma}$ and $\partial_{\alpha\gamma}$:

i.e. $f_{\alpha\beta\gamma}((\partial_{\alpha\beta} x) \otimes y) + f_{\alpha\beta\gamma}(x \otimes (\partial_{\beta\gamma} y)) = \partial_{\alpha\gamma}(f_{\alpha\beta\gamma}(x \otimes y))$



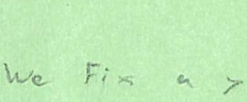
$\mathcal{M}(\psi)$ = 1-dim'l mfd if $\mu(\psi) = 1$, Can break up in the following ways.

$\rightsquigarrow$ Hold y fixed and assume $\partial_{\beta\gamma} y = 0$, since $Y_{\beta\gamma} = \#(g-1)(S^1 \times S^2)$

So then $f_{\alpha\beta\gamma}((\partial_{\alpha\beta} x) \otimes y) = \partial_{\alpha\gamma}(f_{\alpha\beta\gamma}(x \otimes y))$, and

$f_{\alpha\beta\gamma} : \widehat{CF}(\alpha, \beta) \to \widehat{CF}(\alpha, \gamma)$ is a chain map:

We fix a y: Use the top-dim'l cycle in $\widehat{HF}(\#_{g-1}(S^1 \times S^2))$



Called

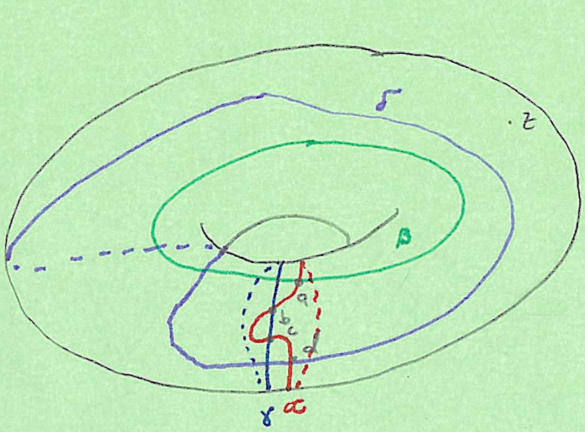
$\theta_{\alpha\beta}$, $\theta_{\alpha\gamma}$, $\theta_{\beta\gamma}$, etc.

We claim the corresponding chain maps

$$\begin{tikzcd} \widehat{CF}(\alpha,\beta) \arrow[rr] & & \widehat{CF}(\alpha,\gamma) \arrow[dl] \\ & \widehat{CF}(\alpha,\delta) \arrow[ul] & \end{tikzcd}$$

induce the exact triangle.

<u>Example</u> $S^3_{-1}(U)$

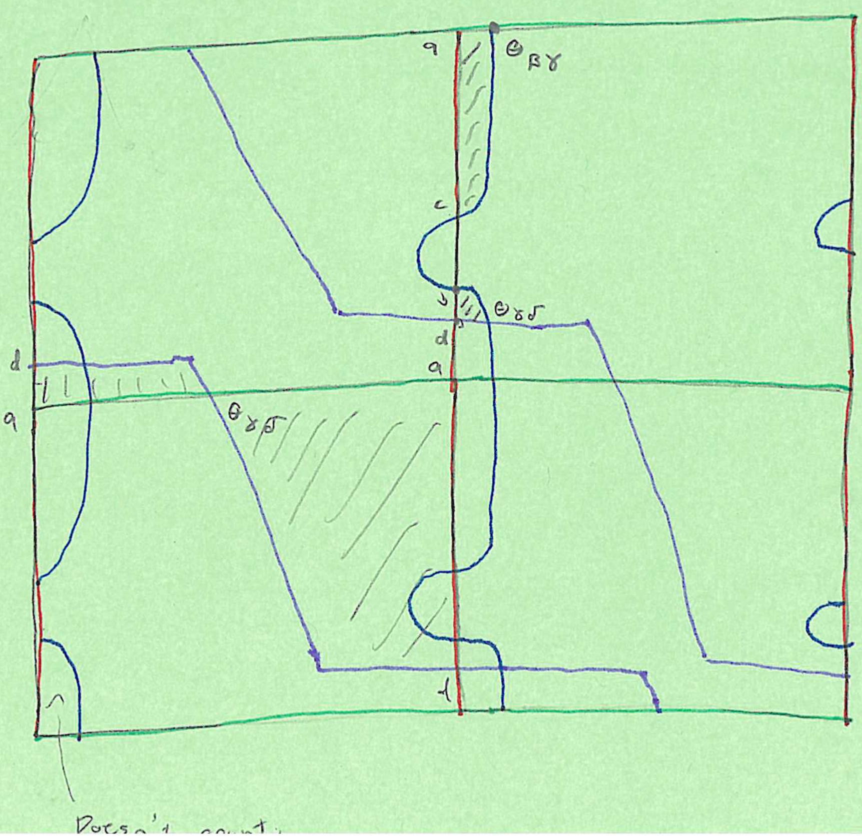


$$\begin{tikzcd} & a \arrow[r, mapsto] & c & & \\ & \widehat{CF}(\alpha,\beta) \arrow[rr] & & \widehat{CF}(\alpha,\gamma) \arrow[dl] & c \arrow[d, mapsto] \quad b \arrow[d, mapsto] \\ a & & \widehat{CF}(\alpha,\delta) \arrow[ul] & & a \quad d \\ d \arrow[u, mapsto] & & & & \end{tikzcd}$$

$\{a\} = \mathbb{T}_\alpha \cap \mathbb{T}_\beta$

$\{b, c\} = \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$

$\{d\} = \mathbb{T}_\alpha \cap \mathbb{T}_\delta$



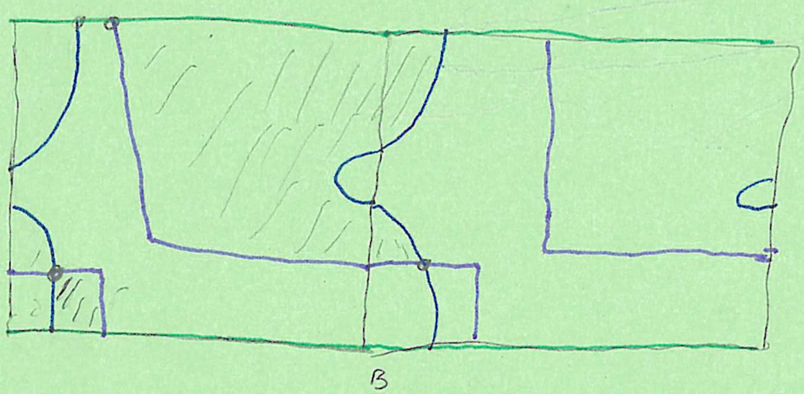
Doesn't count

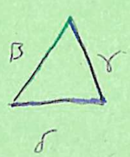
Step Two   we have

$$\begin{tikzcd}
\widehat{HF}(Y) \arrow[rr, "F"] & & \widehat{HF}(Y_0) \arrow[dl, "F_0"] \\
& \widehat{HF}(Y_1) \arrow[ul, "F_1"] &
\end{tikzcd}$$

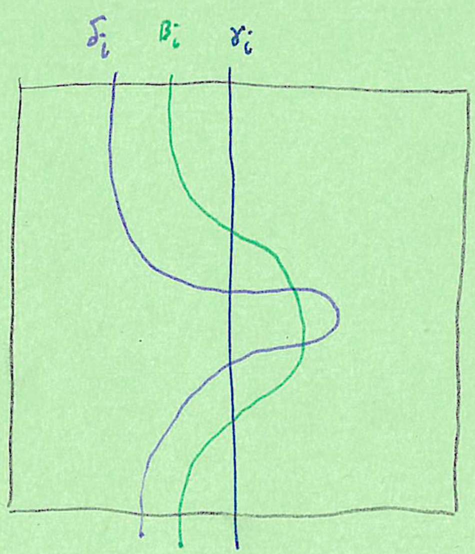
$$F_0 \circ F(\xi) = F_{\alpha\gamma\delta}\left(F_{\alpha\beta\gamma}(\xi \otimes \theta_{\beta\gamma}) \otimes \theta_{\gamma\delta}\right)$$

$$= \hat{F}_{\alpha\beta\delta}\Big(\xi \otimes \underbrace{\hat{F}_{\beta\gamma\delta}(\theta_{\beta\gamma} \otimes \theta_{\gamma\delta})}_{\text{claim: This is zero.}}\Big)$$





And also



Proving this is actually exact

Step One The maps F on homology satisfy an associativity property:

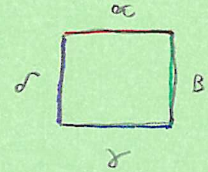
i.e. there is a map $\hat{h}_{\alpha\beta\gamma\delta} : \widehat{CF}(Y_{\alpha\beta}) \otimes \widehat{CF}(Y_{\beta\gamma}) \otimes \widehat{CF}(Y_{\gamma\delta}) \to \widehat{CF}(Y_{\alpha\delta})$

which gives a chain homotopy between

$$\hat{F}_{\alpha\gamma\delta}(F_{\alpha\beta\gamma}(\cdot \otimes \cdot) \otimes \cdot) \;\&\; \hat{F}_{\alpha\beta\delta}(\cdot \otimes F_{\beta\gamma\delta}(\cdot \otimes \cdot))$$

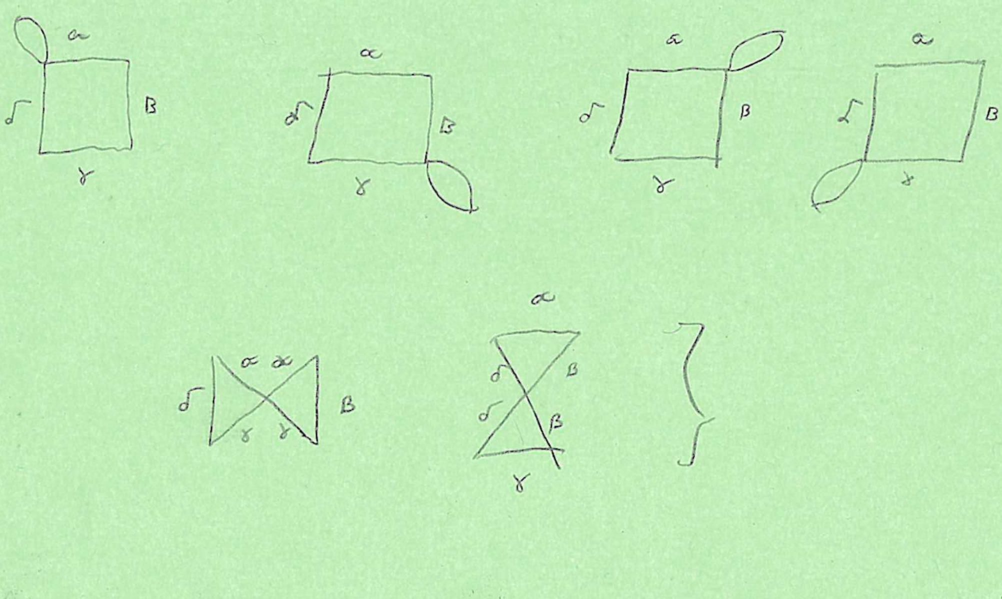
i.e. on homology this is an associative product.

How? We let the map h count rectangles



$$h_{\alpha\beta\gamma}(x \otimes y \otimes w) = \sum_{\substack{p \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \\ \{\phi \in \pi_2(x,y,w,p) : \mu(\phi) = -1,\ n_z(\phi) = 0\}}} \# m(\phi) \cdot p$$

This can break up in six ways:



$\Rightarrow$ $\partial h_{\alpha\beta\gamma\delta} + \hat{h}_{\alpha\beta\gamma\delta}\,\partial = F_{\alpha\gamma\delta}(F_{\alpha\beta\gamma}(\cdot \otimes \cdot) \otimes \cdot) + F_{\alpha\beta\gamma}(\cdot \otimes F_{\beta\gamma\delta}(\cdot \otimes \cdot))$