Relationship of $\tilde{HF}_K$ to $\Delta_K$

A general note

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<td>$\mathbb{N}$</td>
<td>$\dim$ Vector spaces</td>
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<tr>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{K}$ Graded vector spaces</td>
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<td>$\mathbb{Z}[[x, x^{-1}]]$</td>
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<td>$\mathcal{X}(M)$</td>
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<td>$A_K(t)$</td>
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<td>$V_K(t)$</td>
<td>$K^h(K)$</td>
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Claim: $\sum_{i,j} \chi(\tilde{HF}_K(k_{ij})) e^i = \sum_{i,j} \dim(\tilde{HF}_K(k_{ij})) e^i = \Delta_K(t)$

Why?

Kaufman states

We look at a projection and forget (briefly the crossing data). We mark one edge with a basepoint.

A Kaufman state is a map that associates to each double point $v_i$ one of the four corners in such a way that we use each region of $S^2 - S^1$ once.

$\tilde{c} = (c_1, \ldots, c_n)$
To a crossing we associate

\[ \begin{align*}
&\begin{array}{c}
\text{Fig.}
\end{array}
\end{align*} \]

And \( \theta(c_i) \)

\[ \begin{align*}
&\begin{array}{c}
\text{Fig.}
\end{array}
\end{align*} \]

\[ \sum_{c \in K} \prod_{i=1}^{n} (-1)^{\theta(c_i)} \theta(c_i) \text{ (symmetrized)} \]

is the Alexander polynomial of \( K \).

**Exercise** Check that this is true.

- \( A_K(0) = 1 \)
- \( A_L^{-} - A_L^{+} = (t^{-1/2} - t^{1/2}) \Delta(L_0) \)

Why relevant? The Kauffman states correspond to generators in the

pringle-chip diagram.

\[ \begin{align*}
&\begin{array}{c}
\text{Fig.}
\end{array}
\end{align*} \]

\[ \begin{align*}
\text{Thm} \quad A(x) &= \sum_{i=1}^{n} a_i(c_i) \\
M(x) &= \sum_{i=1}^{n} b_i(c_i)
\end{align*} \]
Why is this true?

Two states are said to differ by a transposition if there is a pair of vertices $v_1$ and $v_2$ such that

$G - v_1, v_2 = G - v_2, v_1$.

There is a path $P$ from $v_1$ to $v_2$. Following the knot so that $x(v_1)$ and $y(v_1)$ coincide, any two states can be connected by a transposition.

Kaufmcan

$\sum a(c_i) = \frac{1}{2} \cdot \frac{1}{2} = 0$

$\sum b(c_i) = 1$

$\sum a(c_i) = \frac{1}{2} \cdot \frac{1}{2} = 1$

$\sum b(c_i) = 2$

Exercise: $m = 1$

$[x(\mathcal{K}) = \frac{k}{4}, \frac{l}{4}] + p_x(q) \cdot p_y(p) = 0 - 1 + 2$
What does this mean for alternating knots?

If \( \Delta_k(t) = \sum_{i=-n}^{n} a_i t^i \), \( \sigma(k) \) its signature, then

\[
\widehat{\text{HF}}_{\text{c}}(S^3, k, j) = \begin{cases} 
\mathbb{Z} & \text{if } (i, j) = (i + \frac{\sigma(k)}{2}, j), \\
0 & \text{otherwise}
\end{cases}
\]

\[\text{Corollary } t(k) = \frac{\sigma(k)}{2}\]

Visualization: Two options you'll see.

\[\widehat{\text{HF}}_{\text{c}}(S^3, 3, j)\]

\[\Delta_k(t) = t - 1 + t^{-1}\]

Q: How does this change if we go to the right-handed trefoil?

Q: What is the relationship between \( \Delta_k(t) \) and \( t(k) \)?
While we're on the topic:

Recall

- breadth \( D_k(t) \) \( \leq 2g(k) \)
- \( k \) fibred \( \Rightarrow D_k(t) \) monic

We have

\[ \text{Thm} \quad \max S_j : \dim \overline{HF}^k (K, j) \neq 0 \quad \Rightarrow \quad g(K) = \] [Oesau-th-Szabo]

\[ \text{Thm} \quad k \text{ fibred} (\Rightarrow) \]

Example

\[ D_k(t) = 1 \]
Example 9.

What is $\hat{H}\mathbb{F}_2$? What is $\mathbb{Z}$?

What happens if I go to the mirror? Why?

More specifically, what is an Alexander grading?

A spin$^c$ structure on $S^3 - K$. Note that $\Sigma_2^c(\hat{S}^3 - K) \cong \text{Spin}^c(S^2 - K) \cong H^2(S^3 - K)$

$\sim \mathbb{Z}$

$A(x) = \frac{1}{2} \left< \chi(\Sigma_2^c(\hat{S}^3 - K)), [F] \right>$