Relationship of $\widehat{HFK}$ to Δ_K

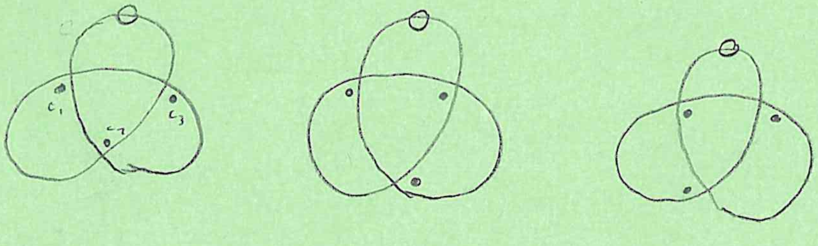
A general note

object	Categorification
$\mathbb{N}$	$\xleftarrow{\dim}$ Vector spaces
$\mathbb{Z}$	$\xleftarrow{\chi}$ Graded vector spaces
$\mathbb{Z}[t, t^{-1}]$	Bigraded vector spaces
$\chi(M)$	$H^*(M)$
$\Delta_K(t)$	$\widehat{HFK}(K)$
$V_K(t)$	$Kh(K)$

Claim $$\sum_{i,j} \chi(\widehat{HFK}(K,j))\, t^j = \sum_{i,j} (-1)^i \dim(\widehat{HFK}_i(K,j))\, t^j = \Delta_K(t)$$

Why?

Kauffman states: We look at a projection and forget (briefly) the crossing data. We mark one edge w/ a basepoint.



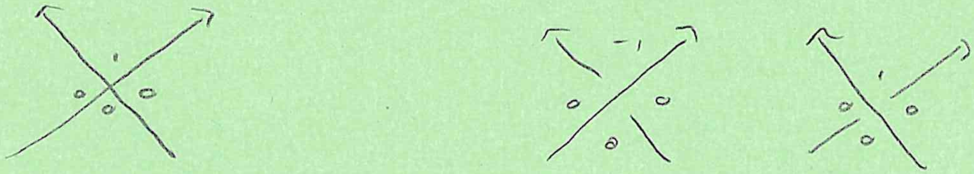
A Kauffman state is a map that associates to each double point v_i one of the four corners in such a way that we use each region of $S^2 - S^1$ exactly once.

$\vec{c} = (c_1, \dots, c_n)$

To a crossing we associate

$a(c_i)$

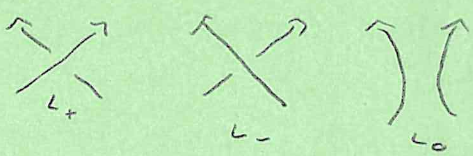
And $B(c_i)$



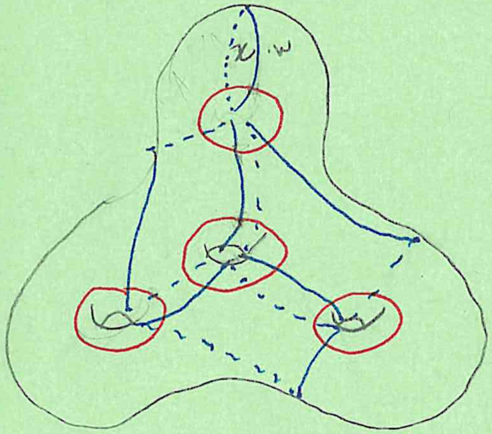
$$\sum_{c \in K} \prod_{i=1}^{n} (-1)^{B(c_i)} T^{a(c_i)}$$ is the (symmetrized) Alexander polynomial of K.

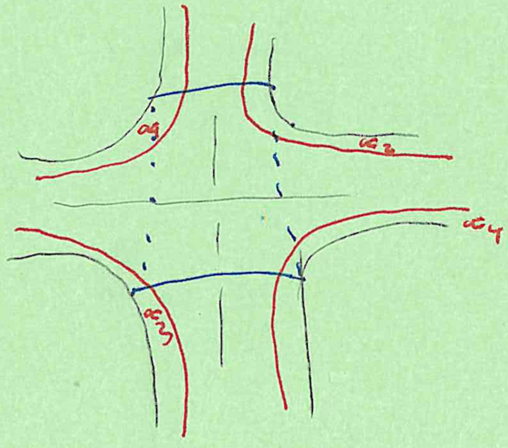
<u>Exercise</u> Check that this is true.

- $\Delta_K(0) = 1$
- $\Delta(L_+) - \Delta(L_-) = (t^{1/2} - t^{-1/2})\,\Delta(L_0)$



Why relevant? The Kauffman states correspond to generators in the pringle-chip diagram.





} Pick a corner

<u>Thm</u> $A(\vec{x}) = \sum_{i=1}^{n} a(c_i) \qquad M(\vec{x}) = \sum_{i=1}^{n} b(c_i)$

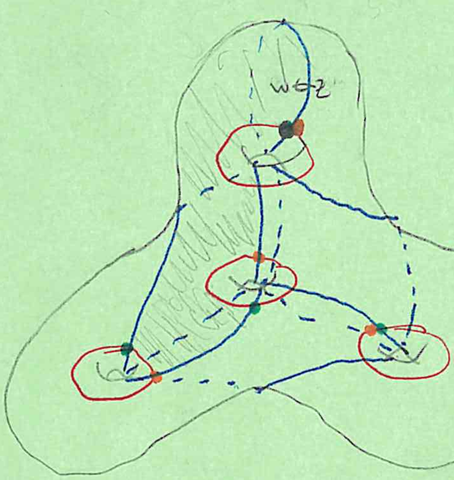
Why is this true?

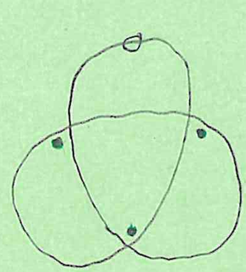
- Two states x and y are said to differ by a transposition if there is a pair of vertices v_1 and v_2 st

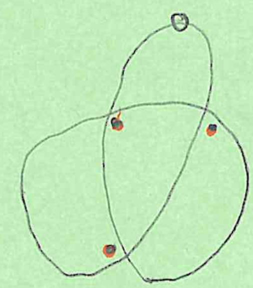
$$x|_{G - v_1 - v_2} = y = |_{G - v_1 - v_2}$$

- There is a path P from v_1 to v_2 following the knot so that $x(v_1)$ and $y(v_1)$

<u>Kauffman</u> Any two states can be connected by a transposition.

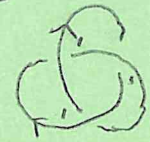






a)

b)

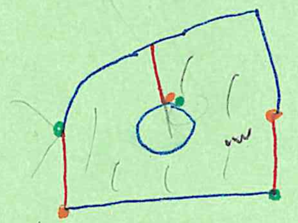


$\sum a(c_i) = \frac{1}{2} + \frac{-1}{2}$
$= 0$

$\sum b(c_i) = 1$

$\sum a(c_i) = \frac{1}{2} + \frac{1}{2}$
$= 1$

$\sum b(c_i) = 2$



$\Big\}$ $\cdot\ \cdot = -1$

Disk goes from $\cdot$ to $\cdot$.

$\Big\}$ $A(\cdot) - A(\cdot) = -1$

$M(\cdot) - M(\cdot) = 1 - 2(1) = -1$

Exercise: $w = 1$

$$\left[x(x) - \frac{k}{4}, \frac{l}{4} \right] + p_x(D) \cdot p_y(D) = \underbrace{0 - 1}\ + 2$$

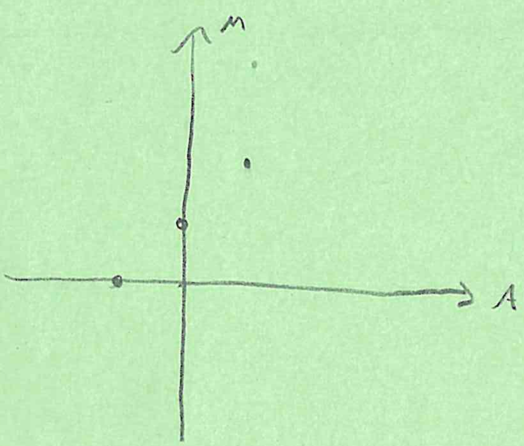
What does this mean for alternating knots?

If $\Delta_K(T) = \sum_{j=-n}^{n} a_j T^j$, $\sigma(K)$ its signature, then

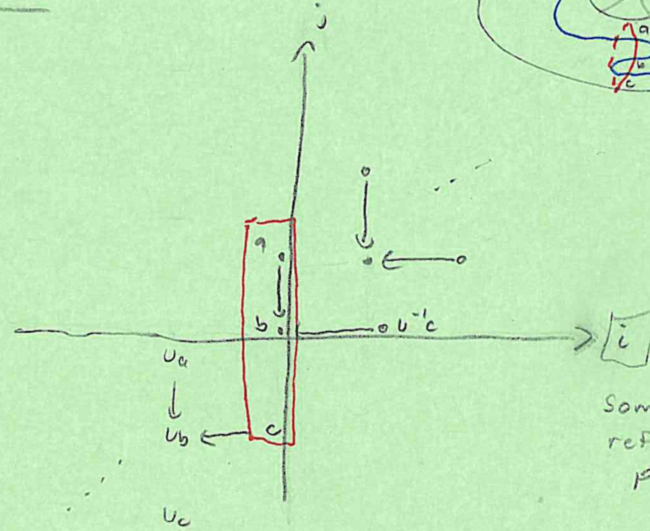
$$\widehat{HFK}_i(S^3, K, j) = \begin{cases} \mathbb{Z}_2^{|a_j|} & (i, j) = \left(j + \frac{\sigma(K)}{2}, j\right), \\ 0 & \text{otherwise} \end{cases}$$

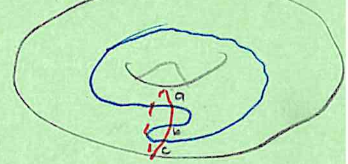
<u>Corollary</u> $\tau(K) = \frac{\sigma(K)}{2}$

<u>Visualization: Two options you'll see</u>



$\widehat{HFK}(S^3, 3_1)$





Sometimes refers to power of U.

$CFK^\infty(S^3, 3_1)$

$\Delta_K(t) = t - 1 + t^{-1}$

Q: How does this change if we go to the right-handed trefoil?

Q: What is the relationship btwn

While we're on the topic:

Recall

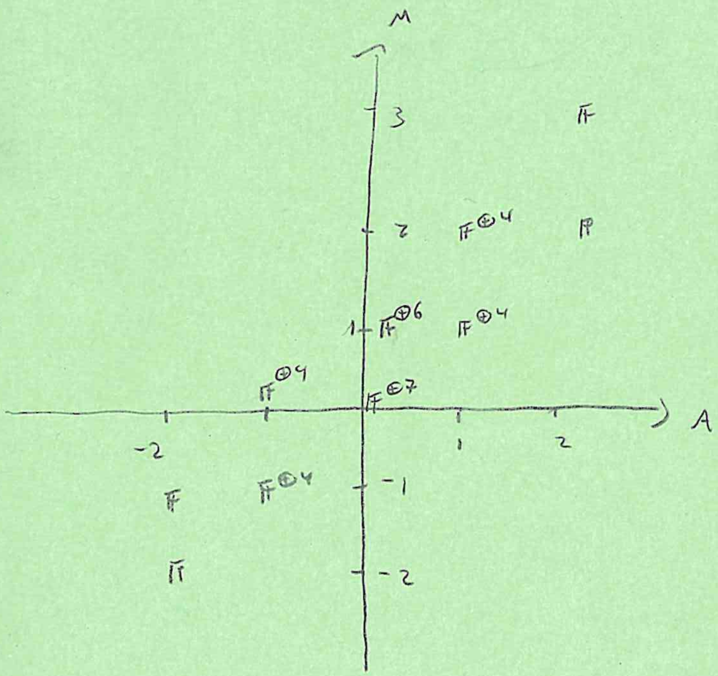
- $\operatorname{breadth}(\Delta_K(t)) \le 2g(K)$
- K Fibred $\Rightarrow$ $\Delta_K(t)$ monic

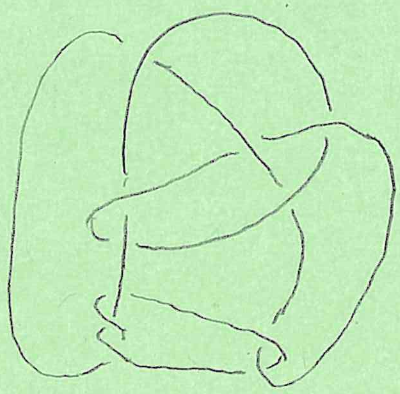
We have

Thm $\max\{j : \dim \widehat{HFK}(K,j) \neq 0\} = g(K)$ [Ozsváth-Szabó]

Thm K Fibred $\Leftrightarrow$

Example

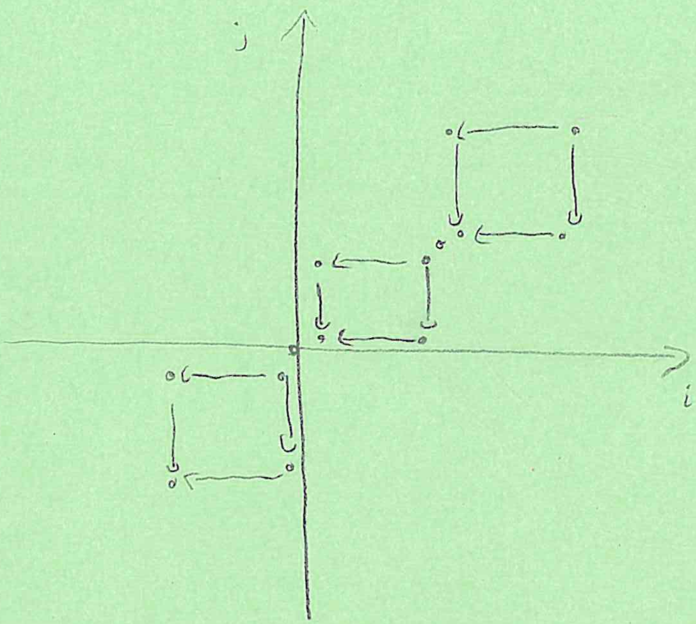




Not Fibred, genus two.

$\Delta_K(t) = 1$

Example 4,



- What is $\widehat{HFK}$? What is τ?
- What happens if I go to the mirror? Why?

More specifically, what is an Alexander grading?

A spin^c-structure on $S^3 - K$. Note that $\underline{s}_{z,w}(\mathbf{x}) \in Spin^c(S^3 - K) \cong H^2(S^3 - K)$
$\cong \mathbb{Z}$

$$A(\mathbf{x}) = \frac{1}{2}\left(c(\underline{s}_{z,w}(\mathbf{x}), [\hat{F}] \right)$$