Lecture 5

Last Time $\mathcal{H} = (\Sigma, \vec{\alpha}, \vec{\beta}, z) \rightsquigarrow CF^{\circ}(Y, s)$

$\mathcal{H} = (\Sigma, \vec{\alpha}, \vec{\beta}, z, w) \rightsquigarrow CFK^{\circ}(Y, s)$

Today Invariance

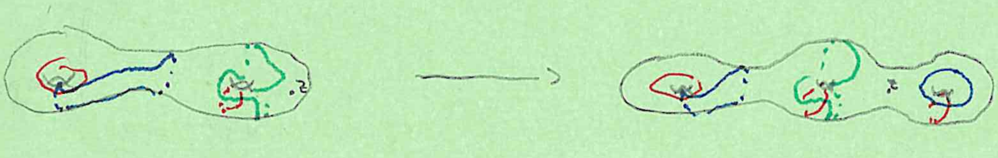
Gradings

Relationship of $\widehat{HFK}$ to Δ_K

Claim Suppose $\mathcal{H} = (\Sigma, \vec{\alpha}, \vec{\beta}, z)$ and $\mathcal{H}' = (\Sigma', \vec{\alpha}', \vec{\beta}', z')$ are two Heegaard diagrams for Y. Then there is a chain homotopy equivalence $\Phi(\mathcal{H}, \mathcal{H}') : CF^{\circ}(\mathcal{H}, s) \longrightarrow CF^{\circ}(\mathcal{H}, s)$.

How are these maps constructed?

Easiest Case: Stabilization



$(Sym^{g}(\Sigma), \mathbb{T}_{\vec{\alpha}}, \mathbb{T}_{\vec{\beta}}) \hookrightarrow (Sym^{g+1}(\Sigma'), \mathbb{T}_{\vec{\alpha}'}, \mathbb{T}_{\vec{\beta}'}))$

- $\mathbb{T}_{\vec{\alpha}} \cap \mathbb{T}_{\vec{\beta}} \simeq \mathbb{T}_{\vec{\alpha}'} \cap \mathbb{T}_{\vec{\beta}'}$
- Disks degenerate over the boundary region.
- Special case of connected sums $\rightsquigarrow$ thm.

Isotopies and changes to cpx structure

- We're now working inside a single symplectic manifold
- One can show that this reduces to changing the cpx structure/metric on the mfd & to a sequence of Hamiltonian isotopies.

Handleslides

- Special case of maps associated to cobordisms (which is wednesday's lecture).

Gradings

Heegaard Floer is naturally $\mathbb{Z}/2\mathbb{Z}$-graded

- $\mathrm{Sym}^g(\Sigma)$ has an orientation induced from Σ
- Give $\mathbb{T}_{\vec{\alpha}}, \mathbb{T}_{\vec{\beta}}$ orientations
- $\vec{x} \in \mathbb{T}_{\vec{\alpha}} \cap \mathbb{T}_{\vec{\beta}} \rightsquigarrow \iota(\vec{x})$ is the nbr s.t. the orientation on $T_{\vec{x}}(\mathrm{Sym}^g(\Sigma))$ is $\iota(\vec{x}) \in \{\pm 1\}$ times the orientation on $T_{\vec{x}}(\mathbb{T}_{\vec{\alpha}}) \cap T_{\vec{x}}(\mathbb{T}_{\vec{\beta}})$.

- We get (up to sign) the algebraic intersection number of $\mathbb{T}_{\vec{\alpha}}$ and $\mathbb{T}_{\vec{\beta}}$

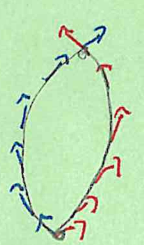
  $\rightsquigarrow$ if we get the orientations on $\mathbb{T}_{\vec{\alpha}}$ and $\mathbb{T}_{\vec{\beta}}$ via orienting the curves, this is the determinant of their intersection matrix.

Exercise

The boundary operator for the decomposition of Y into a 0-cell, 1-cells a_i, 2-cells b_i and a 3-cell is

$$\partial b_i = \sum_{i=1}^{g} \#(\alpha_i \cap \beta_j)\, a_i \qquad \text{(i.e. uses the matrix above)}$$

Given $\phi \in \pi_2(\vec{x}, \vec{y})$, we can check $\iota(\vec{x}) \cdot \iota(\vec{y}) = (-1)^{\mu(\phi)}$.



So we have $\widehat{HF}(Y) = \bigoplus_{i \in \mathbb{Z}/2\mathbb{Z}} \widehat{HF}_i(Y)$, where the boundary operator changes the grading.

If Y is a rational homology sphere, then
$\rightsquigarrow \chi(\widehat{HF}(Y)) = \pm|H_1(Y;\mathbb{Z})|$

$$\begin{aligned} &\chi(\widehat{HF}(Y)) = \pm|H_1(Y;\mathbb{Z})| \\ &\quad \| \\ &\chi(\widehat{CF}(Y)) \\ &\quad \| \\ &\# \,\Pi_{\vec{\alpha}} \cap \Pi_{\vec{\beta}}. \end{aligned}$$

If $H_1(Y;\mathbb{Z})$ is not finite then $\chi(\widehat{HF}(Y)) = 0$.

In fact for $s \in Spin^c(Y)$,

$$\chi(\widehat{HF}(Y,s)) = \begin{cases} \pm 1 & \text{if } H_1(Y;\mathbb{Z}) \text{ finite} \\ 0 & \text{otherwise} \end{cases}$$

Pf of the first thing: $\widehat{HF}(Y,s)$ must be independent of s, since we can change the identification of the intersection pts w/ spin^c-structures by changing the basepoint (and the differential doesn't matter to χ).

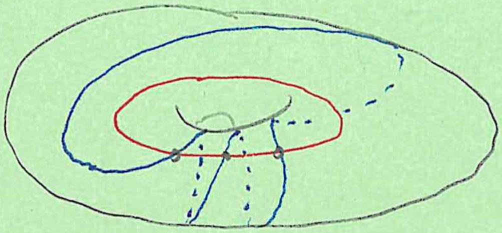
Aside

Defn An L-space is a rational homology sphere for which any of the following (equivalent) things are true:

- $\widehat{HF}(Y)$ is a free abelian w/ rank $|H^2(Y;\mathbb{Z})|$
- $HF^-(Y)$ is a free $\mathbb{Z}[U]$-module w/ rank $|H^2(Y;\mathbb{Z})|$
- $HF^\infty(Y)$ is a free $\mathbb{Z}[U,U^{-1}]$ module of rank $|H^2(Y;\mathbb{Z})|$, and $U: HF^+(Y) \longrightarrow HF^+(Y)$.

The difference between a $\mathbb{Z}$-L-space and a $\mathbb{Z}/2\mathbb{Z}$-L-space is an open question that won't be relevant to any of our examples.

Modelled on lens spaces



- For a $\mathbb{Q}HS^3$, $gr(\vec{x}, \vec{y}) = \mu(\phi) - 2n_z(\phi)$ is well-defined (exercise) and we can use it to get a relative $\mathbb{Z}$-grading.
- Using cobordism maps this can in fact be lifted to an absolute $\mathbb{Q}$-grading.
- Example

$$CF^+(\Sigma(2,3,5)) = \begin{array}{c} \vdots \\ \bullet\, 6 \\ \downarrow \\ \bullet\, 4 \\ \downarrow \\ \bullet\, 2 \end{array} \qquad CF^+(S^3) = \begin{array}{c} \vdots \\ \downarrow \\ \bullet\, 4 \\ \downarrow \\ \bullet\, 2 \\ \downarrow \\ \bullet\, 0 \end{array}$$

Note that this implies $HF^+(\Sigma(2,3,5) \# -\Sigma(2,3,5)) = HF^+(S^3)$.

Open After accounting for this issue, does HF detect the three-sphere? (I.e. is an irreducible Y w/ $HF^+(Y) = \begin{array}{c} \vdots \\ \bullet\, 4 \\ \downarrow \\ \bullet\, 2 \\ \downarrow \\ \bullet\, 0 \end{array}$ necessarily S^3?)

Open Are S^3 and $\pm\Sigma(2,3,5)$ the only $\mathbb{Z}HS^3$ L-spaces?

Gradings on $CFK^-(\mathcal{H})$

Say we have $\mathcal{H} = (\Sigma, \vec{\alpha}, \vec{\beta}, w, z)$ For (S^3, K)

We have the maslov grading:

$gr(\vec{x}, \vec{y}) = \mu(\phi) - \underbrace{2n_w(\phi)}_{\text{Stupid issue here}}$] For $\phi \in \pi_2(\vec{x}, \vec{y})$

There's only one spin^c-structure so this always exists

Can lift to an absolute grading via $\widehat{G:HFK}(S^3, K) \longrightarrow \widehat{HF}(S^3) = \mathbb{F}_{(0)}$. ($G = M$)

Usually we let $i = M(\vec{x})$] Differs from "Introduction"!!!

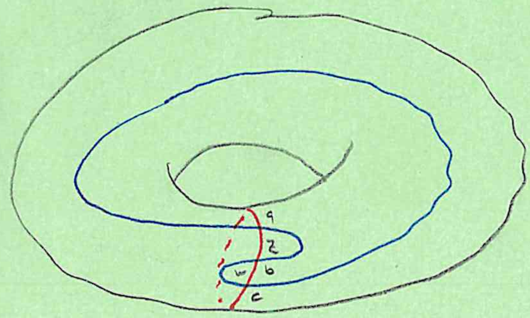
We also have an <u>Alexander grading</u>

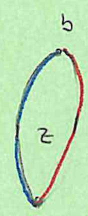
<u>Relatively this is</u> $F(\vec{x}, \vec{y}) = n_z(\phi) - n_w(\phi)$ For $\phi \in \pi_2(\vec{x}, \vec{y})$.

<u>Exercise</u> This lifts uniquely to a Function $A = F : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \longrightarrow \mathbb{Z}$

w/ the property that $\#\{\vec{x} \in \mathbb{T}_{\vec{\alpha}} \cap \mathbb{T}_{\vec{\beta}} : A(x) = j\} = \#\{x \in \mathbb{T}_{\vec{\alpha}} \cap \mathbb{T}_{\vec{\beta}} : A(x) = -j\}$ (mod 2)

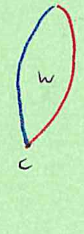
<u>Example</u>





$A(a) - A(b) = 1$

$M(a) - M(b) = 1$



$A(c) - A(b) = -1$

$M(c) - M(b) = 1 - 2 = -1$

a (2, 1)

b (1, 0)

$\boxed{c}$ (0, -1)

The Alexander grading that survives is a (concordance)

Observe that the Alexander grading is preserved by the differential for $\widehat{CFK}$.

- For minus: we set $A(U\hat{x}) = A(x) - n$ and then we have the same result.

Relationship of $\widehat{HFK}$ to Δ_K

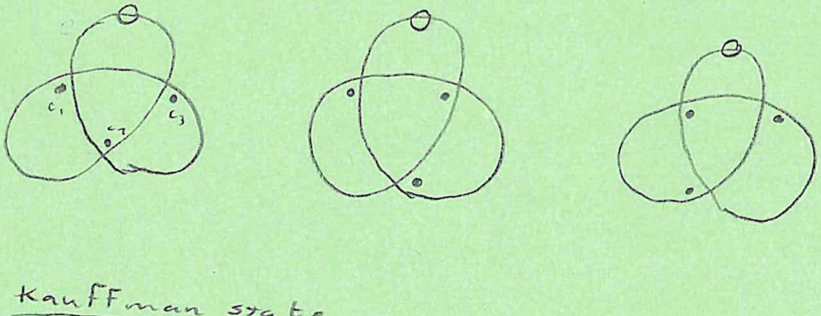
A general note

Object		Categorification
$\mathbb{N}$	$\xleftarrow{\dim}$	Vector spaces
$\mathbb{Z}$	$\xleftarrow{\chi}$	Graded vector spaces
$\mathbb{Z}[t, t^{-1}]$		Bigraded vector spaces
$\chi(M)$		$H^*(M)$
$\Delta_K(t)$		$\widehat{HFK}(K)$
$V_K(t)$		$Kh(K)$

<u>Claim</u> $\sum_{i,j} \chi(\widehat{HFK}(K,j))\, t^j = \sum_{i,j} (-1)^i \dim(\widehat{HFK}_i(K,j))\, t^j = \Delta_K(t)$

<u>Why?</u>

<u>Kauffman states</u> We look at a projection and forget (briefly the crossing data. We mark one edge w/ a basepoint.



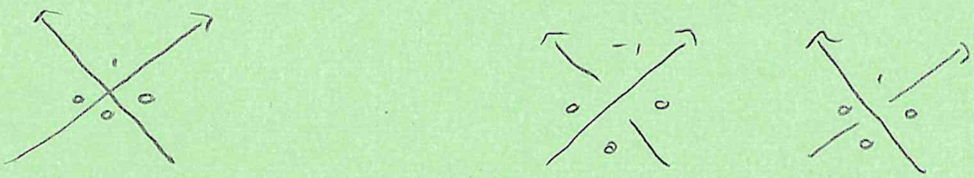
A <u>Kauffman state</u> is a map that associates to each double point v_i one of the four corners in such a way that we use each region of $S^2 - S^1$ exactly once.

$\vec{c} = (c_1, \ldots, c_n)$

To a crossing we associate

$a(c_i)$

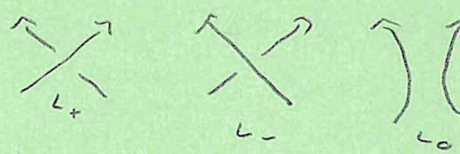
And $b(c_i)$



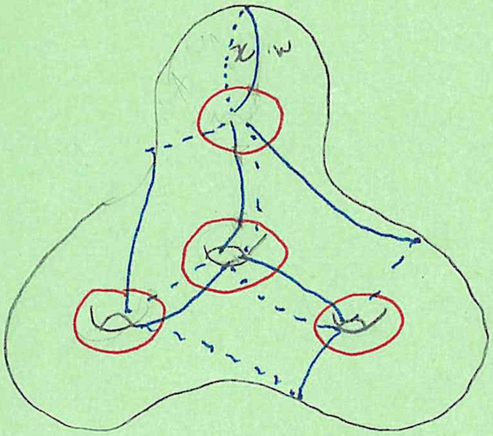
$\sum_{c \in K} \prod_{i=1}^{n} (-1)^{b(c_i)} T^{a(c_i)}$ is the (symmetrized) Alexander polynomial of K.

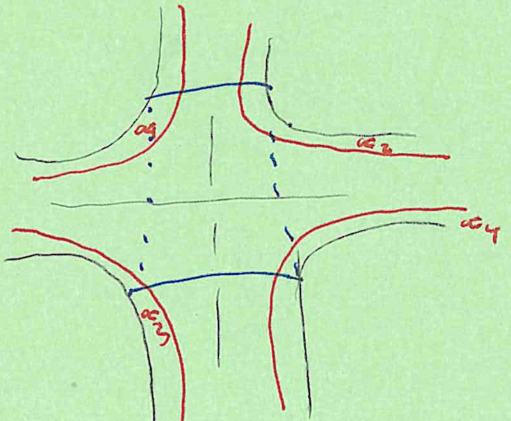
Exercise Check that this is true.

- $\Delta_K(0) = 1$
- $\Delta(L_+) - \Delta(L_-) = (t^{1/2} - t^{-1/2})\Delta(L_0)$



Why relevant? The Kauffman states correspond to generators in the pringle-chip diagram.





Pick a corner

Thm $A(\vec{x}) = \sum_{i=1}^{n} a(c_i)$ $\quad M(\vec{x}) = \sum_{i=1}^{n} b(c_i)$

What does this mean for alternating knots?

If $\Delta_K(T) = \sum_{j=-n}^{n} a_j T^j$, $\sigma(K)$ its signature, then 2

$$\widehat{HFK}_i(S^3, K, j) = \begin{cases} \mathbb{Z}_2^{|a_i|} & (i,j) = \left(i + \frac{\sigma(K)}{2}, j\right), \\ 0 & \text{otherwise} \end{cases}$$