Last Time \[ H = (\Sigma, \tilde{a}, \tilde{b}, z) \mapsto \mathcal{CF}^0(\gamma, s) \]
\[ H = (\Sigma, \tilde{a}, \tilde{b}, z, w) \mapsto \mathcal{CF}^0(\gamma, s) \]

Today

Invariance

Gradings

Relationship of $\mathcal{HFK}$ to $A_k$

Claim Suppose $H = (\Sigma, \tilde{a}, \tilde{b}, z)$ and $H' = (\Sigma', \tilde{a}', \tilde{b}', z')$ are two Heegaard diagrams for $Y$. Then there is a chain homotopy equivalence $\Phi (H, H') : \mathcal{CF}^0(H, s) \rightarrow \mathcal{CF}^0(H', s)$.

How are these maps constructed?

Easiest Case: Stabilization

(\text{Sym}^3(\Sigma, \pi_a, \pi_B)) \quad \rightarrow \quad (\text{Sym}^3(\Sigma', \tilde{\pi}_a, \tilde{\pi}_B'))

Isotopies and Changes to epr Structure

We're now working inside a single symplectic manifold.

One can show that this reduces to changing the epr structure/metric on the $\text{mfld}$ to a sequence of Hamiltonian isotopies.
Handle slides

- Special case of maps associated to cobordisms (which is Wednesday's lecture).

Grading: Heegaard Floer is naturally $Z/2Z$-graded

- $\text{Sym}^3(\Sigma)$ has an orientation induced from $\Sigma$.
- Give $\pi_0$, $\pi_1$ orientations.
- $x \in \pi_0 \cap \pi_1$ means $\iota(x)$ is the 1st or the orientation
  on $T_x^\Sigma(\text{Sym}^3(\Sigma))$ is $(\iota(x) \cdot \pm 1)$ times the orientation
  on $T_x^\Sigma(\pi_0) \cap T_x^\Sigma(\pi_1)$.

  We get (up to sign) the algebraic intersection number
  of $\pi_0$ and $\pi_1$.

  And if we get the orientations on $\pi_0$ and $\pi_1$
  via orienting the curves, this is the determinant
  of their intersection matrix.

Exercise: The boundary operator For the decomposition of $\Sigma$
into a 0-cell 1-cells $a_i$, 2-cells $b_i$ and a 3-cell $c$

$$\partial c = \sum_{i=1}^{g} \# (\iota_i(\partial b_i)) a_i$$
(i.e. uses the matrix above)

Given $Z_{\pi_1}(\Sigma, \gamma)$, we can check $\iota(\Sigma, \gamma) = (\gamma) + (\partial)$.

So we have $\text{HF}(\Sigma) = \bigoplus \text{HF}_i(\Sigma)$ where the boundary
operator changes the grading.
If $Y$ is a rational homology sphere, then

$$\chi(\hat{HF}(Y)) = \pm |H_1(Y; \mathbb{Z})|$$

$$\chi(GF(Y))$$

$$\pi = \pi^+ \wedge \pi^-.$$

If $H_1(Y; \mathbb{Z})$ is not finite then $\chi(\hat{HF}(Y)) = 0.$

In fact for $\text{Spin}^c(Y),$

$$\chi(\hat{HF}(Y; s)) = \begin{cases} 1 & \text{if } H_1(Y; \mathbb{Z}) \text{ finite} \\ 0 & \text{otherwise} \end{cases}$$

**Proof of the First Thing** $\hat{HF}(Y; s)$ must be independent of $\epsilon,$ since we can change the identification of the intersection pts w/ spin$^c$-structures by changing the basepoint (and the differential doesn't matter to $\chi$).

**Aside**

**Definition** An L-space is a rational homology sphere for which any of the following (equivalent) things are true:

- $\hat{HF}(Y)$ is a free abelian w/ rank $|H_2^*(Y; \mathbb{Z})|$
- $HF^-(Y)$ is a free $\mathbb{Z}[u]/u^2$-module w/ rank $|H_2^-(Y; \mathbb{Z})|$
- $HF^{uo}(Y)$ is a free $\mathbb{Z}[u, u^{-1}]$-module of rank $|H_2^*(Y; \mathbb{Z})|$
- $u : H^+(Y) \to H^+(Y)$

The difference between a $\mathbb{Z}$-L-space and a $\mathbb{Z}/2\mathbb{Z}$-L-space is an open question that won't be relevant to any of our examples.
For a $\mathbb{C}H^3$, $g^1(x, y) = m(q^2) - 2n_2(q)$ is well-defined (exercise) and we can use it to get a relative $\mathbb{Z}$-grading.

Using cobordism maps this can in fact be lifted to an absolute $\mathbb{Z}$-grading.

Example

\[
CF^+(\Sigma(2, 3, 5)) = \begin{pmatrix} 6 \\ 4 \\ 2 \\ 0 \end{pmatrix}, \quad CF^+(S^3) = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}
\]

Note that this implies $HF^+(\Sigma(2, 3, 5) \# -5\Sigma(2, 3, 5)) = HF^+(S^3)$.

Open: After accounting for this issue, does $HF^+$ detect the three-sphere? (I.e., is an irreducible $Y$ w/ $HF^+(Y) = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}$ necessarily $S^3$?)

Open: Are $S^3$ and $\Sigma(2, 3, 5)$ the only $\mathbb{C}H^3$ L-spaces?
Graddings on $\mathcal{CFk}^-(\mathcal{K})$

Say we have $\mathcal{K} = (\Sigma, \beta, \mathcal{B}, \nu, c)$. For $(s^3, \kappa)$

We have the master grading:

$$\mathfrak{g}(x, y) = m(x) - 2n_\nu(x)$$

for $x \in \pi_2(s^3, \kappa)$

There's only one spin-c structure so this always exists.

Can lift to an absolute grading via $\varphi : \mathcal{HFk}(s^3, \kappa) \to \mathcal{HF}(s^3) = \mathbb{F}_{\mathbb{Q}}$.

Usually we let $i : M(\mathcal{K})$ differs from "Introduction"!!!

We also have an Alexander grading:

Relatively this is $\mathfrak{f}(x, y) = n_\mathcal{B}(x) - n_\nu(x)$ for $x \in \pi_2(x, y)$.

Exercise: This lifts uniquely to a function $A : F : \pi_{\infty} \times \pi_{\infty} \to \mathbb{Z}$

with the property that $\# \sum_{x \in \pi_{\infty} \times \pi_{\infty}} x \cdot A(x) = \sum_{x \in \pi_{\infty} \times \pi_{\infty}} A(x) = -j \varphi$ \hspace{1cm} (mod 2)

Example

\[ A(a) - A(b) = 1 \]
\[ M(a) - M(b) = 1 \]
\[ A(c) - A(b) = -1 \]
\[ M(c) - M(b) = 1 - 2 = -1 \]

The Alexander grading that survives is a concordance.
Observe that the Alexander grading is preserved by the differential for $\hat{CFK}$.

For minus, we set $A(x^k) = A(x) - m$ and then we have the same result.
### Relationship of \( \widetilde{HF}_k \) to \( A_k \)

**A general note**

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<td>( \mathbb{N} )</td>
<td>( \text{dim} ) vector spaces</td>
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<td>( \mathbb{Z} )</td>
<td>( \mathbb{Z} ) graded vector spaces</td>
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<tr>
<td>( \mathbb{Z}[t, t^{-1}] )</td>
<td>( \mathbb{Z} )-graded vector spaces</td>
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<td>( \chi(M) )</td>
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<td>( A_k(t) )</td>
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<td>( V_k(t) )</td>
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**Claim**

\[
\sum_{ij} \chi(\widetilde{HF}_k(K_{ij})) t^i = \sum_{ij} (t^i)^{\text{dim}(\widetilde{HF}_k(K_{ij}))} t^i = A_k(t)
\]

**Why?**

**Kaufman states** We look at a projection and forget (briefly the crossing data. We mark one edge with a basepoint.

A **Kaufman state** is a map that associates to each double point \( v \) one of the four corners in such a way that we use each region of \( S^2 - S^1 \) once,

\[
\tilde{e} = (e_1, \ldots, e_n)
\]
To a crossing we associate

\[ a_{(c_i)} \]

And \( \theta(c_i) \)

\[ \sum_{c \in K} \prod_{i=1} \theta(c_i) \quad \text{is the Alexander polynomial of } K. \]

Exercise: Check that this is true.

\[ \cdot \Delta_K(0) = 1 \]

\[ \cdot \Delta(k) - \Delta(l) = (t^{1/2} - t^{-1/2}) \Delta(k) \]

Why relevant? The Kauffman states correspond to generators in the pringle-chip diagram.

Theorem: \( A(\kappa) = \sum_{i=1} \delta(c_i) \) and \( M(\kappa) = \sum_{i=1} \lambda(c_i) \)
What does this mean for alternating knots?

If $\Delta_k(T) = \sum_{i=-n}^{n} q_i T^3$, $\sigma(k)$ its signature, then

$\text{HFK}_1(\mathbb{S}^3, x_1, x_2) = \begin{cases} \mathbb{Z}_{2} & (i, j) = (i + \sigma(k), j), \\ 0 & \text{otherwise}. \end{cases}$