Lecture 4

Last Time For our purposes, a spin^c-structure is a nowhere-vanishing vector field v on Y.

- $v_1 \sim v_2$ if they are homotopic after deleting a ball θ from Y.

- $\{v \text{ nonvanishing vector fields}\} \longleftrightarrow \{\text{maps } F_v : Y \to S^2\}$

$$\longleftrightarrow H^2(Y, \mathbb{Z})$$

after choosing a trivialization

- $v \longmapsto \delta^\tau(v) := F_v^*(\underset{\tau}{\underbrace{\alpha}})$ a generator of $H^2(S^2)$.

- $\delta(v_1, v_2) = \delta^\tau(v_1) - \delta^\tau(v_2) \in H^2(Y, \mathbb{Z})$ independent of trivialization

  - Action of $H^2(Y)$ on $\text{Spin}^c(Y)$ via letting $a \cdot v$ be st $\delta(a \cdot v, v) = a$

- $\text{Spin}^c(Y)$ comes w/ an involution $s \longmapsto \bar{s}$ via $v \longmapsto -v$

- We also have a First Chern class

$$c_1 : \text{Spin}^c(Y) \longrightarrow H^2(Y, \mathbb{Z})$$

$$s \longmapsto s - \bar{s}$$

Back to Heegaard diagrams.

We have a map $s_{\vec{z}} : \pi_{\vec{\alpha}}^{-1} \cap \pi_{\vec{\beta}}^{-1} \longrightarrow \mathrm{Spin}^c(Y)$ which refines ε.

Let
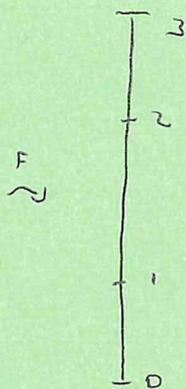
Picking $\vec{x}$ and $\vec{y}$ picks a set of g trajectories from index one to index 2 critical pts. We also have a trajectory from index 0 to index 3. Delete nbhds m) get a complement of disjoint unions on which ∇F does not vanish.

We can fill this in to get a nowhere-vanishing vector field :

Indeed $\underline{\mathrm{Thm}}$ $s_{\vec{z}}(y) - s_{\vec{z}}(x) = PD[\varepsilon(\vec{x}, \vec{y})]$ i.e. $s_{\vec{z}}(y) = s_{\vec{z}}(x) \Leftrightarrow \pi_2(\vec{x}, \vec{y})$ nonempty.
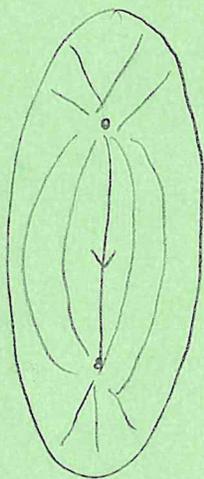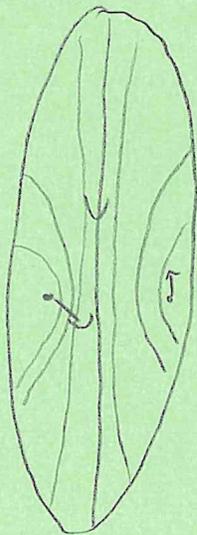
In general one can do something of this ilk for any case where the indices of the critical pts differ.

Boundary operator

· We consider disks for which $\mu(\phi) = 1$.

· For a given homotopy class of disks $\phi \in \pi_2(x,y)$, we let $M(\phi)$ be the moduli space of (pseudo)-hol'c representations of ϕ.

· This comes to us w/ an action of $\mathbb{R}$

translation
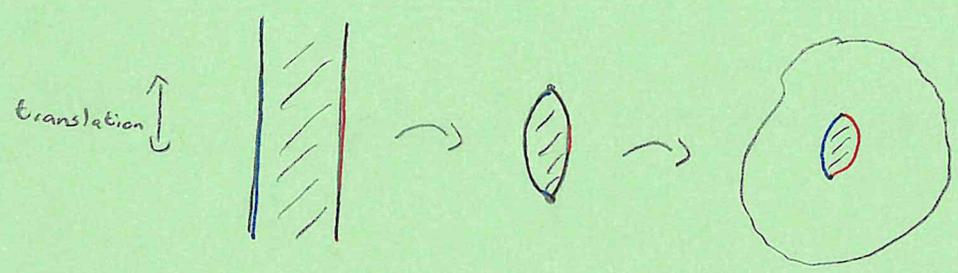
· We quotient out to get rigid pseudohol'c disks

$$\hat{M}(\phi) = \frac{M(\phi)}{\mathbb{R}}$$

## Add one more condition:

We ask for Heegaard diagrams which are <u>weakly admissible</u> i.e. any disk whose shadow's boundary is $\partial D(\phi) = \sum n_i \alpha_i + m_i \beta_i$ contains domains of both positive and negative multiplicities.

A periodic domain

Yes   No
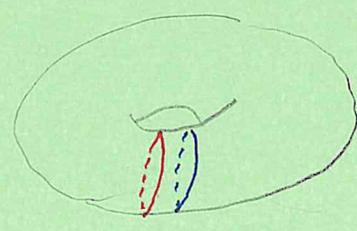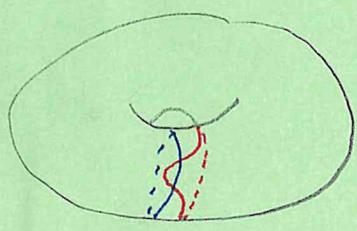
Prop^n There is a family of perturbations of the complex structure with the property that if $u(\phi) = 1$ then $\hat{M}(\phi)$ is a compact oriented zero-dimensional mfd.

**Helpful Note** If we have $u \in M(\phi)$, there is a map

$$S \xrightarrow{\hat{u}} \Sigma$$

g-Fold →
branched
cover
$$\left\downarrow P \qquad \downarrow P \right.$$

$$D^2 \xrightarrow{u} Sym^g(\Sigma)$$

We get a hol'c map

$$\phi_u \quad \hat{u} : S \longrightarrow \Sigma$$
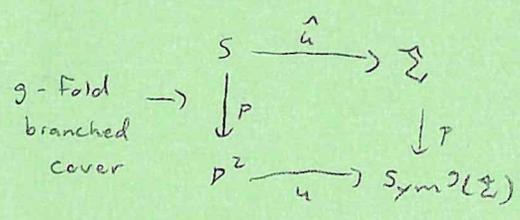
Examples
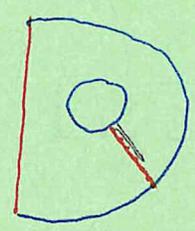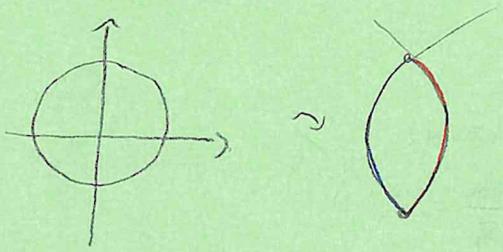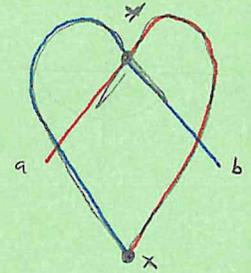
There is a one-dim'l.
Family of maps by
the Riemann mapping
thm; we quotient
and get a 0-dim'l
Family.
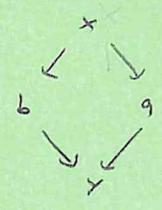
∃ a biholomorphism
between any two
simply-ctd domains
in $\mathbb{C}$ ~) in fact
given that the
space of maps
$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is 3d, fixing
$ad - bc = 1$ two points
gives a 1D set

$u(\phi) = 1$

$u(\phi) = 2$

The boundary operator is

$$\partial x = \sum_{y \in \pi_\alpha \cap \pi_B} \left(\sum_{\substack{\phi \in \pi_2(x,y) \\ \mu(\phi) = 1}} \# \hat{M}(\phi) \cdot U^{n_z(\phi)} y \right)$$

$$\hat{\partial} x = \sum_{y \in \pi_\alpha \wedge \pi_B} \left(\sum_{\substack{\phi \in \pi_2(x,y) \\ \mu(\phi) = 1 \\ n_z(\phi) = 0}} \# \hat{M}(\phi) \, y \right) \quad \left.\begin{array}{l} \text{This is equivalent} \\ \text{to counting disks} \\ \text{in } \mathrm{Sym}^9 (\Sigma \setminus \{z\}) \end{array}\right.$$

In the case $b_1 = 0$.
This comes with a grading: $gr(x,y) = \mu(\phi) - 2n_z(\phi)$ where ϕ is any disk connecting them. Ergo the differential lowers grading by 1.

More generally $\frac{\mathbb{Z}}{2\mathbb{Z}}$-graded

Examples
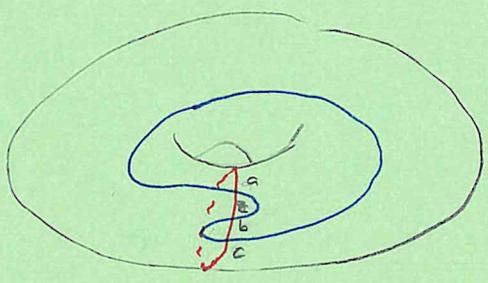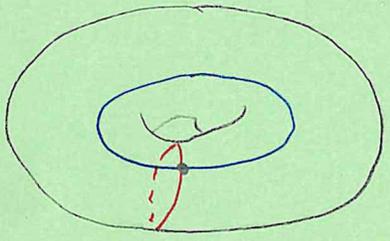
$CF^-(s^3) = \mathbb{F}[U] \langle x \rangle$

$HF^-(s^3) = \mathbb{F}[U]$

$\widehat{HF}(s^3) = \mathbb{F}$

$CF^-(s^3) = \mathbb{F}[U] \langle a, b, c \rangle$

$HF^-(s^3) = \mathbb{F}[U] \langle [U_{c+a}] \rangle$

$\widehat{HF}(s^3) = \mathbb{F} \langle [a] \rangle$
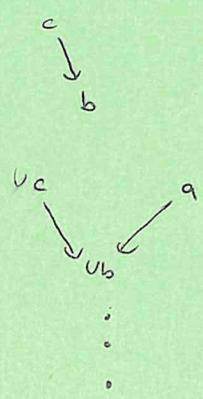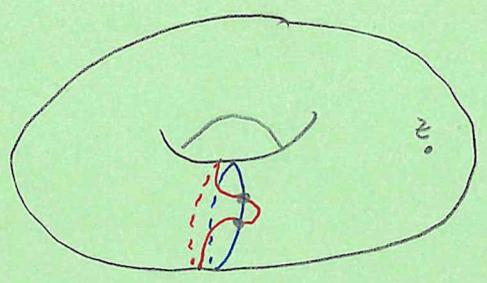
$\bullet\ HF^-(S^3) = \mathbb{F}_2[U]\langle[a],[b]\rangle$

$\cdot$ Every other spin^c structure is empty.

IF we have a diagram for (Y, K), we have a filtration coming from the second basepoint, via counting intersections w/ v_w.

$\bullet\ \partial_K^- \vec{x} = \sum_Y \sum_{\substack{\phi \in \pi_2(\vec{x}, \vec{y}) \\ \mu(\phi)=1 \\ n_w(\phi)=0}} U^{n_z(\phi)} \# \hat{\mathcal{M}}(\phi) \vec{y}$
$\qquad \hat{\partial}_K \vec{x} = \sum_Y \sum_{\substack{\phi \in \pi_2(\vec{x}, \vec{y}) \\ \mu(\phi)=1 \\ n_w(\phi)=n_z(\phi)=1}} \# \hat{\mathcal{M}}(\phi) \vec{y}$

$kFK^-(3_1) = \mathbb{F}[U]\langle a, b, c\rangle$

$HFK^-(3_1) = \mathbb{F}[U]\langle[c]\rangle \oplus \mathbb{F}\langle[b]\rangle$
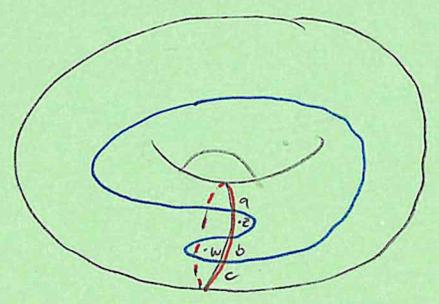
$\widehat{HFK}(3_1) = \mathbb{F}\langle[a],[b],[c]\rangle$

Note
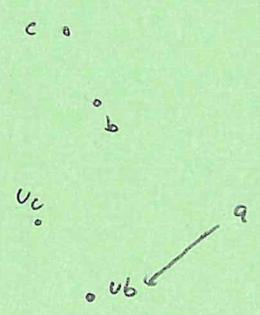
$0 \to CFK^-(Y,K) \xrightarrow{\cdot U} CFK^-(Y,K) \longrightarrow \widehat{CFK}(Y,K) \to 0$

So you expected
$$\widehat{HFK}(3_1) = \ker(U) \oplus \mathrm{coker}(U)$$

what is a $spin^c$ - structure?

$SO(3) = SU(2)/\{\pm 1\} = U(2)/U(1)$

$\| \quad$
$\mathbb{RP}^3$

$\qquad 1 \to \mathbb{Z}_2 \to Spin(3) \to SO(3) \to 1$

$\qquad\qquad\qquad \| $
$\qquad\qquad\qquad SU(2) = S^3$

$\qquad 1 \to \mathbb{Z}_2 \to Spin^c(3) \to SO(3) \times U(1) \to$

$\qquad\qquad\qquad\quad \| $
$\qquad\qquad\qquad \Big(Spin^c(3) \times U(1)\Big)\Big/\{\pm 1\}$

$\qquad\qquad\qquad\quad \| $
$\qquad\qquad\qquad\quad U(2)$

• $U(2)$ principal circle bundle $\Longleftrightarrow$ $[x, BU(1)] = H^2(x)$

$\qquad \downarrow$
$\qquad SO(3)$
$\qquad \underbrace{\qquad\qquad}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad K(\mathbb{Z}, 2)$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathbb{CP}^\infty$

This bundle
corresponds
to the nontrivial element
in $H^2(SO(3); \mathbb{Z}) = \mathbb{Z}_2$

• $Spin(3) = SU(2)$

$Spin_c(3) = \Big(U(1) \times Spin(3)\Big)\Big/\{\pm 1\} = \Big(U(1) \times SU(2)\Big)\Big/\{\pm 1\} = U(2)$

• Y closed oriented 3-mfd, has a metric

$\qquad TY \rightsquigarrow F_r$ principal $SO(3)$ bundle.

$\qquad \downarrow \qquad\quad \downarrow$
$\qquad Y \qquad\quad Y$

• A $spin^c$-structure is a lift of F_r to a principal $Spin^c(3) = U(2)$ bundle

• So a $U(2)$-bundle F w/ an isomorphism $F/U(1) \cong F_r$

$\qquad\qquad\qquad \downarrow \qquad\qquad\qquad\qquad\qquad \downarrow$
$\qquad\qquad\qquad Y \qquad\qquad\qquad\qquad\qquad Y$

$\rightsquigarrow$ A $Spin^c$-structure is an element of $H^2(F_r)$ whose reduction to every fiber is the nonzero element of $H^2(F_r)$. $Spin^c(Y)$

via the element α associated to F.

$$F \downarrow$$
$$F/U(1) \cong F_r$$

· $H^2(Y)$ acts on $H^2(F_r)$ via pullback $H^2(Y) \longrightarrow H^2(F_r)$. This is free and transitive $[F_r = Y \times SO(3)$ after picking a trivialization of TY, so $H^2(F_r) = H^2(M) \oplus (\mathbb{Z}/2\mathbb{Z})$.$]$

We also have the description given previously, which we can rewrite as an element of $H^2(SY)$ whose restriction to every fiber is the oriented generator of $H^2(S^2)$.

$$p: SO(3) \longrightarrow S^2$$
$$(e_1, e_2, e_3) \longmapsto e_1$$

is a circle bundle w/ fiber corresponding to the nonzero element of $H_1(SO(3)) = \mathbb{Z}/2\mathbb{Z}$

Pullback $p^*: H^2(S^2) \longrightarrow H^2(SO(3))$ sends a generator of $H^2(S^2)$ to a generator of $H^2(SO(3))$
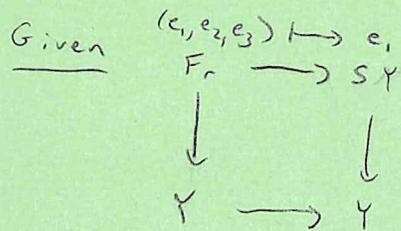
Given $\begin{array}{c} (e_1, e_2, e_3) \longmapsto e_1 \\ F_r \longrightarrow SY \\ \downarrow \qquad \downarrow \\ Y \longrightarrow Y \end{array}$ $\qquad p^*: H^2(SY) \longrightarrow H^2(F_r)$ sends a nonzero vector field to a spin^c-structure.

<u>Note</u>. $TY = u \oplus u^\perp \leadsto$ structure group $(1) \oplus U(1) \subseteq U(2)$ $\Big\}$ of course has a spin^c-structure

· $c_1(s) = c_1(u^\perp)$ is $\delta^v(v) - \delta^v(\cdot v) = s - \bar{s}$