Lecture 3

Macdonald "Symmetric Products of an Algebraic Curve"

Last Time

- $\mathcal{H} = (\Sigma, \vec{\alpha}, \vec{\beta}, z)$ a Heegaard diagram for Y an oriented closed 3-mfd

or

- $\mathcal{H} = (\Sigma, \vec{\alpha}, \vec{\beta}, z, w)$ a Heegaard diagram for (Y, K) an oriented closed 3-mfd w/ a nulhomologous knot.

We consider $Sym^g(\Sigma)$ } n-dim'l complex mfd.

$\pi_1(Sym^g(\Sigma)) \simeq H_1(Sym^g(\Sigma)) \simeq H_1(\Sigma)$

Contains

- $\mathbb{T}_{\vec{\alpha}} = \alpha_1 \times \dots \times \alpha_g$
- $\mathbb{T}_{\vec{\beta}} = \beta_1 \times \dots \times \beta_g$

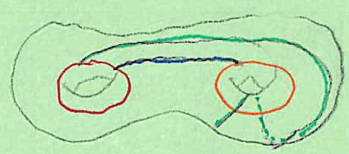
} totally real, $\mathbb{T}_{\vec{\alpha}} \pitchfork \mathbb{T}_{\vec{\beta}}$

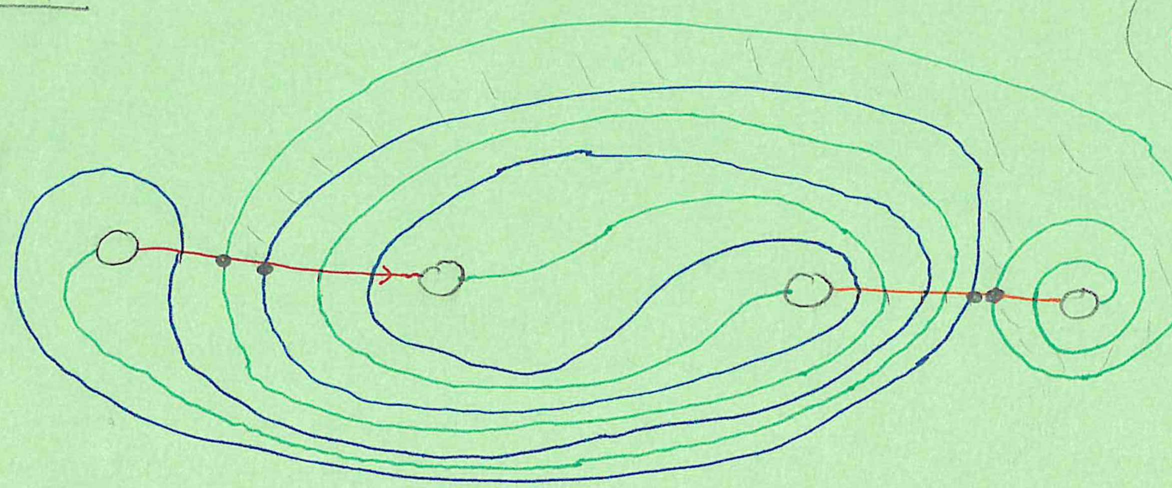
$$\frac{H_1(Sym^g(\Sigma))}{H_1(\mathbb{T}_{\vec{\alpha}}) \oplus H_1(\mathbb{T}_{\vec{\beta}})} \simeq \frac{H_1(\Sigma)}{[\alpha_i], [\beta_j]} \simeq H_1(Y)$$

- $V_z = \{z\} \times Sym^{g-1}(\Sigma)$, likewise V_w if relevant

We look at intersection points between $\mathbb{T}_{\vec{\alpha}}$ and $\mathbb{T}_{\vec{\beta}}$ $\rightsquigarrow$ finitely many because the mfds are cpt and transverse.

Example





	•	•
•	3	2
•	3	4

$\rightsquigarrow$ 18 intersection points.

<u>Defn</u> $CF^-(Y) = \bigoplus_{x \in \mathbb{T}_{\vec\alpha} \cap \mathbb{T}_{\vec\beta}} \mathbb{F}[U]\langle x \rangle$

$\widehat{CF}(Y) = \bigoplus_{x \in \mathbb{T}_{\vec\alpha} \cap \mathbb{T}_{\vec\beta}} \mathbb{F}\langle x \rangle$

<u>More generally</u>

$CF^\infty(Y) = \bigoplus \mathbb{F}[U, U^{-1}]\langle x \rangle$

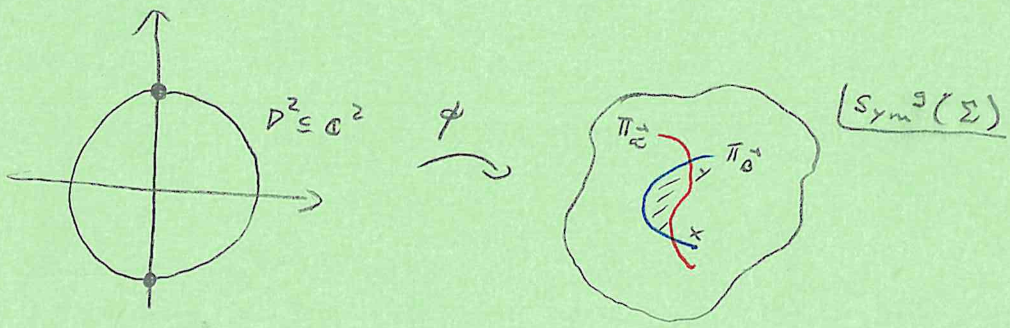
$CF^-(Y)$ is the submodule for which $U^k x$, $k \geq 0$

$CF^+(Y) \simeq CF^\infty(Y) / CF^-(Y)$

$\widehat{CF}(Y) = CF^-(Y) / \langle U = 0 \rangle$

<u>Boundary Operator</u>

<u>Whitney disk</u> A map $\phi : D^2 \longrightarrow Sym^g(\Sigma)$ w/ the following properties.



(Note this is so far just topological.)

- $\pi_2(x, y)$: Set of homotopy classes of such disks.

<u>Example</u>

Disks come w/ a natural multaplicative structure

$\pi_2(x, y) * \pi_2(y, z) \longrightarrow \pi_2(x, z)$

and a splicing

$$\pi_2'(\mathrm{Sym}^g(\Sigma)) * \pi_2(x,y) \longrightarrow \pi_2(x,y)$$

$\uparrow$ basepoint-free

We can also study a disk via studying its <u>shadow</u>.

<u>Defn</u> Let $D_1, \dots, D_m$ be the closures of the cpts of $\Sigma - \{\alpha_1, \dots, \alpha_g\} - \{\beta_1, \dots, \beta_g\}$. Given $\phi \in \pi_2(x,y)$ the shadow of ϕ, or the domain associated to ϕ, is a formal linear combination of the regions $\{D_i\}_{i=1}^m$

$$D(\phi) = \sum_{i=1}^m n_{z_i}(\phi) D_i$$

i.e. we project

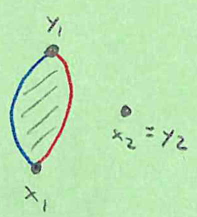
$$\begin{tikzcd} D^2 \arrow[r, hook] \arrow[dr, dashed] & \mathrm{Sym}^g(\Sigma) \arrow[d] \\ & \Sigma \end{tikzcd}$$

. Can also think of this as

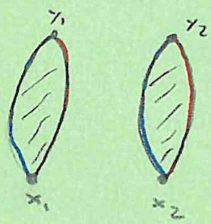
$$\begin{tikzcd} S \arrow[r, hook] \arrow[d] & \Sigma \times \mathrm{Sym}^{g-1}(\Sigma) \arrow[d] \\ D^2 \arrow[r] & \mathrm{Sym}^g(\Sigma) \end{tikzcd}$$

a multiply branched cover of D

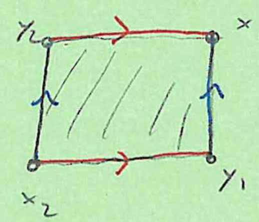
Things that are disks on a diagram of $g=2$ (or greater).



1

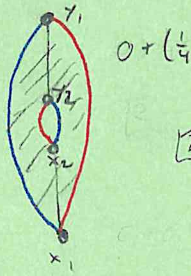


2



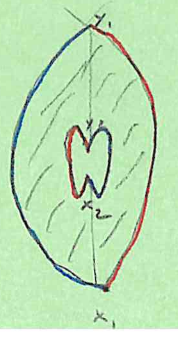
1

$0 + 2(\frac{1}{4}) + 2(\frac{1}{4}) = 1$



$0 + (\frac{1}{4} + \frac{3}{4}) + (\frac{1}{4} + \frac{3}{4})$

$\boxed{2}$



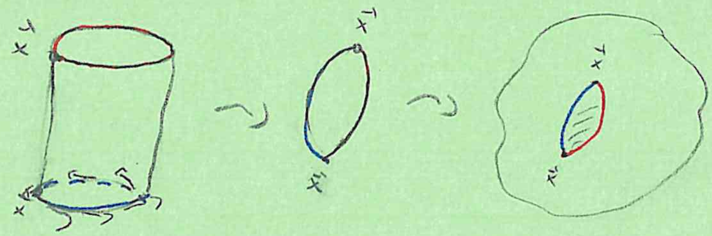
$-1 + (\frac{1}{4}) + (\frac{1}{4}) + (\frac{1}{4}) + (\frac{1}{4})$

$\boxed{0}$

A disk has an associated number called the Maslov index:

- In principle this is a relative chern class of $(\phi^*(TM), \phi^*(T(\mathbb{T}_\alpha)) \oplus \phi^*(T(\mathbb{T}_\beta$

  - Makes sense if $\vec{x} = \vec{y}$



  - Needs some modification for $\vec{x} \neq \vec{y}$.
    - <u>For $g=1$</u> counts # of times $T(\mathbb{T}_{\vec{\alpha}})$ is parallel to $T(\mathbb{T}_{\vec{\beta}})$.
    - More generally, we have a formula

If $D(\phi) = \sum n_i D_i$, then

$$\mu(\phi) = \sum_i n_i e(D_i) + p_{\vec{x}}(D) + p_{\vec{y}}(D)$$

Euler measure of domain; $1 - \frac{k}{2}$ if D a convex $2k$-gon.

Sums of average multiplicities of $\vec{x}$, resp $\vec{y}$ at each corner.

$\left[\text{Also } \chi(S) - \frac{k}{4} + \frac{\ell}{4}\right]$

Acute corners

Obtuse corners

So returning to our examples....

Q Is there a disk connecting any two points of $\mathbb{T}_{\vec{\alpha}} \cap \mathbb{T}_{\vec{\beta}}$?

<u>No, there is an obstruction.</u>

Let $\vec{x}, \vec{y}$ be a pair of intersection points. Let $a: [0,1] \to \mathbb{T}_{\vec{\alpha}}$, $b: [0,1] \to \mathbb{T}_{\vec{\beta}}$ from $\vec{x}$ to $\vec{y}$. Then $a-b$ is a loop in $\mathrm{Sym}^g(\Sigma)$.

We can consider
$$\pi_1(\mathrm{Sym}^g(\Sigma)) \xrightarrow{\sim} H_1(\mathrm{Sym}^g(\Sigma)) \xrightarrow{\sim} H_1(\Sigma)$$
$$a-b \longmapsto \varepsilon(\vec{x},\vec{y})$$

If $\varepsilon(\vec{x},\vec{y}) \neq 0$ then $\pi_2(\vec{x},\vec{y})$ is empty. Also we can calculate $\varepsilon(\vec{x},\vec{y})$ in the surface itself: $a: [0,1] \to \mathbb{T}_{\vec{\alpha}}$ is a collection of arcs on the $\vec{\alpha}$ curves, $b: [0,1] \to \mathbb{T}_{\vec{\beta}}$ is a collection of arcs on the $\vec{\beta}$ curves, and the invariant doesn't care about which choice of arc you take.

<u>Example</u>

- There are two equivalence classes of intersection points on the diagram I drew at the beginning of today.

This links back to our disks:

Let $\vec{x} = \{x_1, \dots x_g\}$ w/ $x_i \in \alpha_i \cap \beta_i$ and $\vec{y} = \{y_1, \dots y_g\}$ w/ $y_i \in \alpha_i \cap \beta_{\sigma^{-1}(i)}$ for σ some suitable permutation. Then given $\phi \in \pi_2(x,y)$, show that

- The restriction of $\partial D(\phi)$ to α_i is a one-chain w/ bdy $y_i - x_i$
- The restriction of $\partial D(\phi)$ to β_i is a one-chain w/ bdy $x_i - y_{\sigma(i)}$.

"$\partial(D(\phi))$ connects $\vec{x}$ to $\vec{y}$ on $\vec{\alpha}$ curves and $\vec{y}$ to $\vec{x}$ on β ..."

iv) If we have $\vec{x}$ and $\vec{y}$ in $\mathbb{T}_{\vec{\alpha}} \cap \mathbb{T}_{\vec{\beta}}$ and $A = \sum_{i=1}^{\hat{n}} a_i D_i$ satisfies that ∂A connects $\vec{x}$ to $\vec{y}$ along $\vec{\alpha}$ curves and $\vec{y}$ to $\vec{x}$ along β curves, then ∂A connects $\vec{x}$ to $\vec{y}$.

<u>Propn</u> For $g > 1$, and $\vec{x}, \vec{y} \in \mathbb{T}_{\vec{\alpha}} \cap \mathbb{T}_{\vec{\beta}}$, if A connects $\vec{x}$ to $\vec{y}$ then there is a class $\phi \in \pi_2(x, y)$ whose shadow is A. If $g > 2$ ϕ is uniquely determined by A.

<u>Propn</u> For $g > 2$, and $\vec{x}, \vec{y} \in \mathbb{T}_{\vec{\alpha}} \cap \mathbb{T}_{\vec{\beta}}$. Then if $\epsilon(\vec{x}, \vec{y}) \neq 0$, $\pi_2(x, y)$ is empty; otherwise $\pi_2(\vec{x}, \vec{y}) \cong \mathbb{Z} \oplus H^1(Y; \mathbb{Z})$.

[The difference here is that $\pi_2(\mathrm{Sym}^2(\Sigma))$ is big and complicated.]

<u>Spin^c-structures</u>

Y closed-oriented 3-mfd.

<u>Defn</u> Let v_1, v_2 nowhere-vanishing vector fields. We say v_1 and v_2 are homologous if there is a ball B in Y st $v_1|_{Y-B}$ is homotopic to $v_2|_{Y-B}$. $\mathrm{Spin}^c(Y)$ is the set of nowhere-vanishing vector fields modulo this relation.

If we fix a trivialization of TY, there is a one-to-one correspondence $\{\text{vector fields } v \text{ over } Y\} \longleftrightarrow \{\text{maps } f_v : Y \to S^2\}$

If u generates $H^2(S^2)$, let $\delta^{\tau}(v) = F_v^*(u)$.

<u>Exercise</u> If v_1 and v_2 are nowhere-vanishing vector fields over Y, then $\delta(v_1, v_2) = \delta^{\tau}(v_1) - \delta^{\tau}(v_2) \in H^2(Y; \mathbb{Z})$ is independent of the trivialization we chose.

$\rightsquigarrow$ This gives an action of $H^2(Y)$ on $\mathrm{Spin}^c(Y)$ via letting $a + v$ be the vector field st $\delta(a+v, v) = a$.

$\rightsquigarrow$ $\mathrm{Spin}^c(Y)$ comes w/ an involution $s \longmapsto \bar{s}$, by taking v to $-v$.

$\rightsquigarrow$ we also have the first Chern class

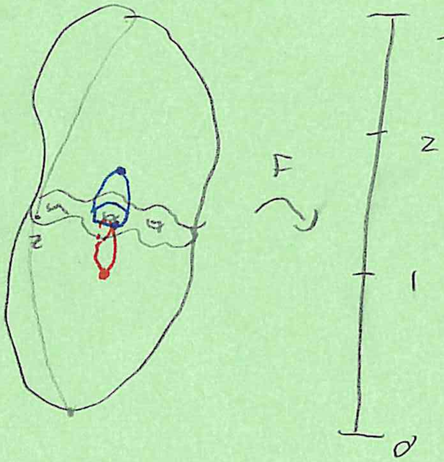
$$c_1 : \mathrm{Spin}^c(Y) \longrightarrow H^2(Y; \mathbb{Z})$$
$$s \longmapsto s - \bar{s}.$$

[In particular, this is the first Chern class of the cpx bundle $v^{\perp}$]

<u>Back to Heegaard diagrams</u>

We have a map $s_z : \mathbb{T}_{\vec{\alpha}} \cap \mathbb{T}_{\vec{\beta}} \longrightarrow \mathrm{Spin}^c(Y)$ which refines ε.

Let,



Picking $\vec{x}$ and $\vec{y}$ picks a set of g trajectories from index one critical pts to index 2 critical pts. We also have a trajectory from the index 0 to index 3 critical pt from the basepoint. Delete nbhds $\rightsquigarrow$ get a complement of disjoint unions on which ∇F does not vanish.

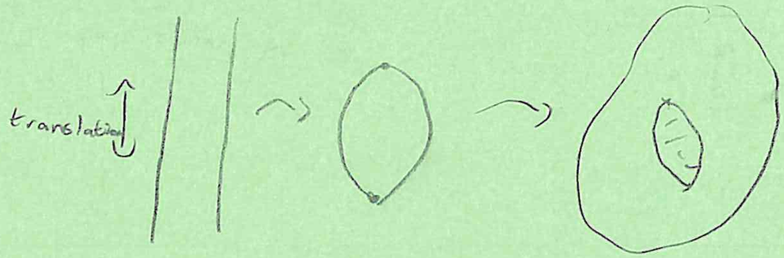
Gradient vector field has index zero and can be extended to a nowhere-vanishing field over Y. This gives a spin^c-structure

Indeed $s_z(y) - s_z(x) = PD[\varepsilon(\vec{x}, \vec{y})]$

i.e. $s_z(y) = s_z(x) \iff \pi_2(\vec{x}, \vec{y})$ nonempty.

Boundary Operator

- We consider disks for which $\mu(\phi) = 1$.
- For a given homotopy class of disks $\phi \in \pi_2(x, y)$, we let $\mathcal{M}(\phi)$ be the moduli space of (pseudo)-holomorphic representatives of ϕ.
- This comes w/ an action of $\mathbb{R}$



We quotient out to get rigid pseudohol'c disks $\hat{\mathcal{M}}(\phi) = \dfrac{\mathcal{M}(\phi)}{\mathbb{R}}$

The boundary operator is

$$\partial^- x = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \left(\sum_{\substack{\phi \in \pi_2(x,y) \\ \mu(\phi)=1}} \# \hat{\mathcal{M}}(\phi) \cdot U^{n_z(\phi)} \, y \right)$$

$$\hat{\partial} x = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \left(\sum_{\substack{\phi \in \pi_2(x,y) \\ \mu(\phi)=1 \\ n_z(\phi)=0}} \# \hat{\mathcal{M}}(\phi)\, y \right) \Bigg\}$$

This is the same as working in $\mathrm{Sym}^g(\Sigma \setminus \{z\})$.