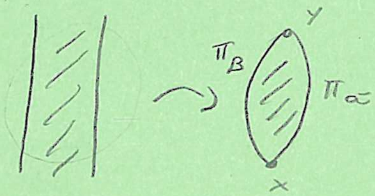
Equivariance Issues and HF

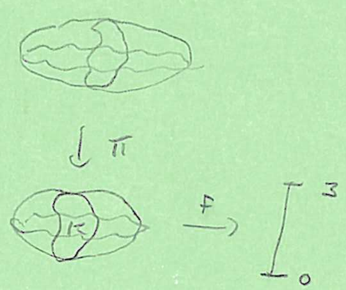
Quick Reminder

- $\widehat{HF}$ is the Lagrangian Floer homology of $\mathbb{T}_\alpha, \mathbb{T}_\beta$ in $Sym^g(\Sigma)$.



Any topological disk has a Maslov index $\mu(\phi)$.

- If we have (eg) a double-branched cover, we can construct a Heegaard diagram w/ a symmetry.



- What good does this do for us?

Equivariant Theory (Just $\mathbb{Z}/2\mathbb{Z}$)

- $X \circlearrowleft \tau$ topological space, $\tau^2 = Id$
- Consider $X \times_{\mathbb{Z}_2} \underset{S^\infty}{\underset{\uparrow}{E\mathbb{Z}_2}} = X \times E\mathbb{Z}_2 /$ (free diagonal action)] Borel construction
- $H^*_{\mathbb{Z}_2}(X) = H^*(X \times_{\mathbb{Z}_2} E\mathbb{Z}_2; \mathbb{F}_2)$

$\exists$ a Fibration

$$\begin{tikzcd} X \arrow[r] & X \times_{\mathbb{Z}_2} E\mathbb{Z}_2 \arrow[d] \\ & pt \times_{\mathbb{Z}_2} E\mathbb{Z}_2 = B\mathbb{Z}_2 = \mathbb{R}P^\infty \end{tikzcd}$$

$\rightsquigarrow H^*_{\mathbb{Z}_2}(X)$ is a module over $H^*(\mathbb{R}P^\infty; \mathbb{F}_2) = \mathbb{F}_2[\theta]$.

<u>Examples</u> • $H^*_{\mathbb{Z}_2}(pt) = H^*(B\mathbb{Z}_2; \mathbb{F}_2) = \mathbb{F}_2[\theta]$

• If $\mathbb{Z}_2 \circlearrowright X$ freely, $H^*_{\mathbb{Z}_2}(X) = H^*(X/\mathbb{Z}_2)$ (A)

<u>Note</u> If $X \longrightarrow Y$ $\mathbb{Z}_2$-equivariant htpy equivalence, then we $H^*_{\mathbb{Z}_2}(X) \xrightarrow{\sim} H^*_{\mathbb{Z}_2}(Y)$. [Inverse need not be equivalent.] (B)

<u>Exercise</u> (A) and (B) determine the theory.

<u>More algebraically</u> we have $C_*(X) \circlearrowleft \tau_\#$ i.e. C_* is an $\mathbb{F}_2[\mathbb{Z}_2]$ chain cpx.

• $H^*_{\mathbb{Z}_2}(X) = \mathrm{Ext}_{\mathbb{F}_2[\mathbb{Z}_2]}(C_*, \mathbb{F}_2)$.
Trivial $\mathbb{F}_2[\mathbb{Z}_2]$-module.

<u>Unpack this</u>

Free resolution of C_* over $\mathbb{F}_2[\mathbb{Z}_2]$:

$$0 \leftarrow C_* \otimes \mathbb{F}_2[\mathbb{Z}_2] \xleftarrow{1+\tau_*} C_* \otimes \mathbb{F}_2[\mathbb{Z}_2] \xleftarrow{1+\tau_*} C_* \otimes \mathbb{F}_2[\mathbb{Z}_2] \longleftarrow \cdots$$

Take Hom to $\mathbb{F}_2$:

$$0 \longrightarrow C^* \xrightarrow{1+\tau^*} C^* \xrightarrow{1+\tau^*} C^* \longrightarrow \cdots$$

] $H^*_{\mathbb{Z}_2}(X)$ is the homology of this complex

Note $\exists$ a spectral sequence $H^*(X;\mathbb{F}_2)\otimes \mathbb{F}_2[\theta] \Rightarrow H^*_{\mathbb{Z}_2}(X)$

$$\mathbb{F}_2[\theta] = H^*(\mathbb{RP}^\infty;\mathbb{F}_2)$$

This is the first page of the Serre ss of $X \to X \times_{\mathbb{Z}_2} E\mathbb{Z}_2 \to B\mathbb{Z}_2$.

Exercise This generalizes the Borel construction: $C_*(X \times E\mathbb{Z}_2)$ is a free resolution of $C_*(X)$ over $\mathbb{F}_2[\mathbb{Z}_2]$.

Let X^{Fix} be the fixed set of τ. If X has the homotopy type of a finite dim'l CW cpx, there is a localization isomorphism:

$$\theta^{-1}H^*_{\mathbb{Z}_2}(X) \longrightarrow H^*(X^{Fix};\mathbb{F}_2)\otimes \mathbb{F}_2[\theta,\theta^{-1}]$$

$\therefore$ a rank inequality

$$\dim(H^*(X;\mathbb{F}_2)) \geq \dim(H^*(X^{Fix};\mathbb{F}_2))$$

(P. Smith 1938, but this formulation due to Borel)

Exercise This follows from consideing the spectral sequence where we start with the $1+\tau_*$ differentials. Uses crucially that if $\sigma \in X^{Fix}$, then $\partial\sigma \in X^{Fix}$.

So we have $H^*(X;\mathbb{F}_2)\otimes \mathbb{F}_2[\theta,\theta^{-1}] \Rightarrow H^*(X^{Fix};\mathbb{F}_2)\otimes \mathbb{F}_2[\theta,\theta^{-1}]$.

Note Everything but localization depends on the qi class of C_*.

Note Finite-dimensionality (as an equivariant CW cpx) is important:

$(x_1, x_2, \ldots) \longrightarrow (x_1, -x_2, -x_3, \ldots)$

④

<u>Note</u> No splitting along homological grading.

<u>Exercise</u> Compute the spectral sequence for  $(x_1, x_2, x_3) \downarrow (-x_1, -x_2, -x_3)$

<u>Let's try to do this in Lagrangian Floer</u>

- $(M, L_0, L_1) \circlearrowleft \tau$ symplectic involution, $\tau^* \omega = \omega$.
- $\tau^2 = \mathrm{Id}$ preserves L_0, L_1 setwise.
- Fixed sets $M^{Fix} \subseteq M$, $L_0^{Fix} \subseteq L_0$, $L_1^{Fix} \subseteq L_1$

<u>Exercise</u> If M^{Fix} connected, show M^{Fix} is symplectic and L_0^{Fix}, L_1^{Fix} are Lagrangians inside it.

<u>Q1</u> When/how can we define $HF_{\mathbb{Z}_2}(L_0, L_1)$?

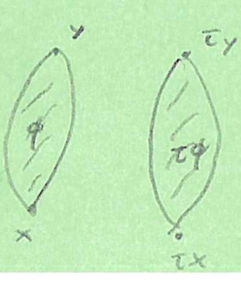
<u>Q2</u> When/how can we relate $HF_{\mathbb{Z}_2}(L_0, L_1)$ to $HF(L_0^{Fix}, L_1^{Fix})$?

<u>Want</u> $CF(L_0, L_1) \circlearrowleft \tau_\#$ w/ $\tau_\#$ a chain map.

$\rightsquigarrow \mathrm{Ext}_{\mathbb{F}_2[\mathbb{Z}_2]}(CF(L_0, L_1), \mathbb{F}_2)$

$\rightsquigarrow$ everything <u>but</u> localization follows.

<u>Plan 1</u> Find $J(t)$ equivariant achieving transversality $\rightsquigarrow$ $\tau_\#$ a chain map



<u>Issue</u>: This is hard! Equivariant cpx structures are not generic. Indeed, in some cases impossible.

eg ϕ:  $\to M^{Fix} \hookrightarrow M$ $\quad \mu_{Fix}(\phi) > \mu(\phi)$] Problem.

Khovanov – Seidel: Such a J can always be found if there are no disks contained in the fixed set.

Seidel-Smith's localization theory

Thm (Seidel-Smith 2010) Under strict K-theoretic assumptions, $\exists$ an equivariant Hamiltonian isotopy of L_0, L_1 fixing L_0^{fix} and L_1^{fix} to L_0', L_1' and $J(t)$ equivariant achieving transversality for (L_0', L_1').
Furthermore $\exists$ a localization isomorphism $\theta^{-1} HF_{\mathbb{Z}_2}(L_0', L_1') \longrightarrow HF(L_0^{fix}, L_1^{fix}) \otimes \mathbb{F}_2[\theta, \theta^{-1}]$.

What assumptions? Recall the problem had something to do w/ relative K-theory

$$\begin{tikzcd}
NL_0^{fix} \times \{0\} \arrow[r, hook] & NM^{Fix} \times [0,1] \arrow[r] \arrow[d] & NM^{Fix} \arrow[d] \\
NL_1^{fix} \times \{1\} \arrow[ur, hook] & & \\
L_0 \times \{0\} \arrow[r, hook] & M^{Fix} \times [0,1] \arrow[r] & M^{Fix} \subseteq M \\
L_1 \times \{1\} \arrow[ur, hook] & &
\end{tikzcd}$$

Want relative triviality (as a cpx bundle) of

$$\left(NM^{Fix} \times [0,1],\ (NL_0^{fix} \times \{0\}) \otimes \mathbb{C} \amalg (NL_1^{fix} \times \{1\} \otimes \mathbb{C})\right)$$

Lots of spectral sequences

	First Page Related To	∞-page related to
Seidel-Smith	$Kh^{symp}(K)$	$\widehat{HF}(\Sigma(K))$
	$Kh^{symp}(\tilde{K})$	$Kh^{symp}(K)$
	$\widehat{HFK}(\Sigma(K), \tilde{K})$	$\widehat{HFK}(S^3, K)$
.)	$\widehat{HFK}(S^3, \tilde{K})$	$\widehat{HFK}(S^3, K)$
	$HF(\phi^2)$	$HF(\phi)$

Reproved by Seidel in a much broader setting

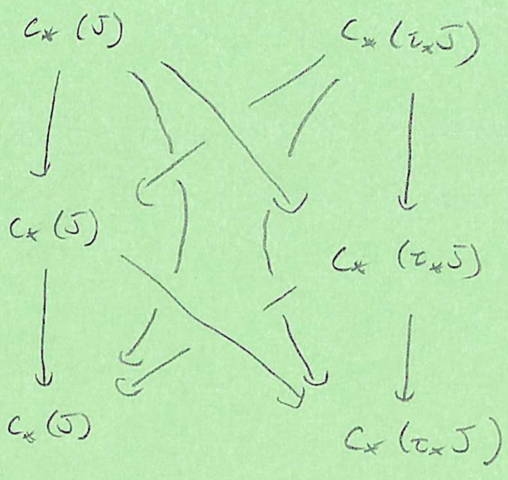
Plan 2 Lots of nonequivariant cpx structures

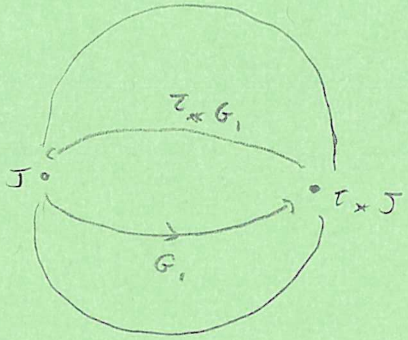
Version 1 (Seidel and Smith) Heavily modeled on Morse theory, more or less an S^∞'s worth of J's.

Version 2 (H.-Lipshitz-Sarkar) A simplicial S^∞'s worth of J's

Version 3 (Bao-Honda) Kuranishi structures, Fairly different

<u>Construction</u> $(M, L_0, L_1) \rightsquigarrow \widetilde{CF}(L_0, L_1)$ the freed Floer complex





$\widetilde{CF}(L_0, L_1) \simeq CF(L_0, L_1)$ as $\mathbb{F}_2$-complexes; $\widetilde{CF}(L_0, L_1)$ is a free $\mathbb{F}_2[\mathbb{Z}_2]$-complex

<u>Thm 1</u> $\exists$ a well-defined $HF_{\mathbb{Z}_2}(L_0, L_1)$ which is an invariant of the equivariant Hamiltonian isotopy type of L_0 & L_1, and a spectral sequence

$$HF(L_0, L_1) \otimes \mathbb{F}_2[\theta] \rightrightarrows HF_{\mathbb{Z}_2}(L_0, L_1)$$

each page of which is similarly an invariant.

<u>Thm 2</u> If $\exists$ an equivariant cpx structure achieving transversality (eg if the S-S conditions are satisfied, or we have a nice diagram) then $(CF(L_0, L_1), \partial_J) \simeq \widetilde{CF}(L_0, L_1)$ as $\mathbb{F}_2[\mathbb{Z}_2]$-complexes. (So every page of the spectral sequences in the table is an invt of the equivariant Hamiltonian isotopy type of L_0 & L_1 & therefore generally the topological input data).

Concordance Invts

Let K be a knot and $\Sigma(K)$ its dbc. There is a well-defined $\mathbb{F}_2[\theta]$-module $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$ w/ the property that $\theta^{-1}\widehat{HF}_{\mathbb{Z}/2\mathbb{Z}}(\Sigma(K))$
$$\simeq \mathbb{F}_2[\theta, \theta^{-1}].$$

Get a spectral sequence $\widehat{HF}(\Sigma(K)) \otimes \mathbb{F}_2[\theta] \rightrightarrows \widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))$.

<u>Defn</u> Let $q_\tau(K)$ be twice the minimal homological grading in $\widehat{HF}_{\mathbb{Z}_2}(\Sigma(K))/\{\text{tors}\}$.

<u>Propn</u> (HLS) q_τ is a concordance invt.

<u>Examples</u> $\Sigma(2,3,7) = \Sigma(T(3,7)) = \Sigma(P(-2,3,7))$

$$\widehat{HF}(\Sigma(2,3,7)) = \begin{array}{ccc} \underset{0}{\bullet} & \overset{?}{\longleftrightarrow} & \underset{0}{\bullet} \\ & & \\ & \bullet\ -1 & \end{array}$$

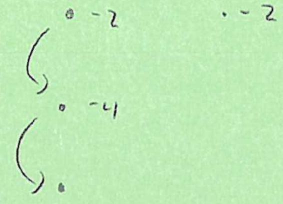
Involution is
- Trivial for $T(3,7)$
- Nontrivial for $P(-2,3,7)$

<u>Exercise</u> This determines the spectral sequence, and implies

$q_\tau(T(3,7)) = 0$

$q_\tau(P(-2,3,7)) = -2$ $\rightsquigarrow$ q_τ is determined by the knot, not the 3-mfd.

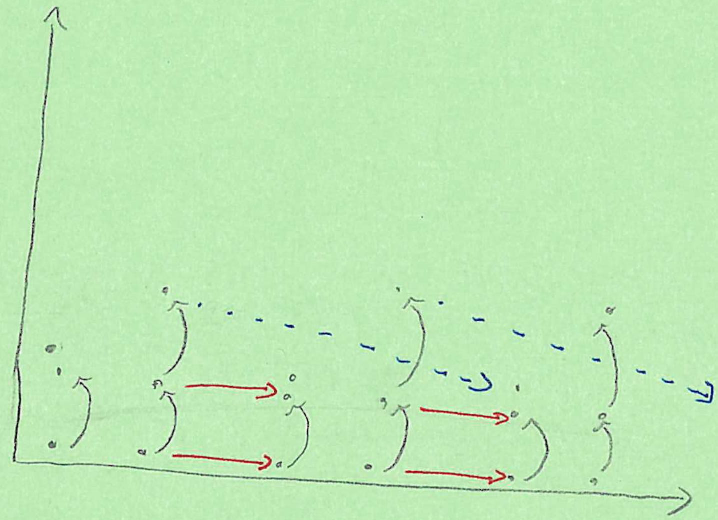
Similarly for HF^-



$HF^-(\Sigma(2,3,7))$

Thm (HLS) $\exists$ an equivariant $HF^-_{\mathbb{Z}_2}(\Sigma(K), s_0)$, which is an $\mathbb{F}[U,\theta]$-module

$\exists$ a spectral sequence w/ $E_1 = HF^-(\Sigma(K), s_0) \otimes \mathbb{F}_2[\theta]$



U-equivariance: Arrows go from tail to U-torsion elements

$$d_{\tilde{\tau}}(K,1) \leq d_{\tilde{\tau}}(K,2) \leq \cdots$$
$$\|$$
$$\tfrac{1}{2}\delta(\Sigma(K), s_0)$$

Concordance invts; not homomorphisms.

Exercise What are $d_{\tilde{\tau}}(K,2)$ of $T(3,7)$, $P(-2,3,7)$, and their mirrors?