Quick Reminder

- \( \hat{HF} \) is the Lagrangian Floer homology of \( \pi_x, \pi_B \) in \( \text{Sym}^g(\Sigma) \).

- Any topological disk has a Maslov index \( m(D) \).

- If we have (eg) a double-branched cover, we can construct a Heegaard diagram with a symmetry.

What good does this do for us?

Equivariant Theory (Just \( \mathbb{Z}_2 \))

- \( X \) is a topological space, \( \mathbb{Z}_2 = \text{Id} \)

  - Consider \( X \times \mathbb{Z}_2 \times E \mathbb{Z}_2 = X \times E \mathbb{Z}_2 \) (free diagonal action)

  - \( H_{\mathbb{Z}_2}^X(X) = H^X \left( X \times \mathbb{Z}_2, E \mathbb{Z}_2 / \mathbb{Z}_2 \right) \)
A fibration $\rightarrow x \times \mathbb{Z}_2 \
\downarrow 
\pt \times \mathbb{Z}_2 = \mathbb{Z}_2 = \mathbb{R}P^\infty$

$H^*_\mathbb{Z}_2(x)$ is a module over $H^*(\mathbb{R}P^\infty, \mathbb{Z}_2) = \mathbb{F}_2[0]$.

Examples: $H^*_\mathbb{Z}_2(pt) = H^*(\mathbb{R}P^\infty, \mathbb{Z}_2) = \mathbb{F}_2[0]$

• If $\mathbb{Z}_2 \vee \times$ freely, $H^*_\mathbb{Z}_2(x) = H^*(x/\mathbb{Z}_2)$ (4)

Note: If $\rightarrow x \mathbb{Z}_2$ - equivariant hopf equivalence, then we

$H^*_\mathbb{Z}_2(x) \sim H^*_\mathbb{Z}_2(y)$. [Inverse need not be equivalent.] (8)

Exercise (8) and (8) determine the theory.

More algebraically we have $c_\ast(x) \otimes \mathbb{F}_2^+$, i.e., $c_\ast$ is an $\mathbb{F}_2[\mathbb{Z}_2]$ chain complex.

$H^*_\mathbb{Z}_2(x) = \text{Ext}_{\mathbb{F}_2[\mathbb{Z}_2]}(c_\ast, \mathbb{F}_2)$, (Trivial $\mathbb{F}_2[\mathbb{Z}_2]$-module.)

Unpack this.

Free resolution of $c_\ast$ over $\mathbb{F}_2[\mathbb{Z}_2]$:

$0 \leftarrow c_\ast \otimes \mathbb{F}_2[\mathbb{Z}_2] \leftarrow \mathbb{Z}_2 \otimes \mathbb{F}_2[\mathbb{Z}_2] \leftarrow \mathbb{Z}_2 \rightarrow c_\ast \otimes \mathbb{F}_2[\mathbb{Z}_2] \leftarrow \cdots$

Take Hom to $\mathbb{H}_2$:

$0 \rightarrow c_\ast \rightarrow c_\ast \rightarrow c_\ast \rightarrow \cdots$ \quad \Rightarrow \quad H^*_\mathbb{Z}_2(x)$ is the cohomology of this complex.
Note 3 a spectral sequence $H^*(x; F_2) \otimes F_2[\theta^{-1}] \Rightarrow H^*_B(x)$

This is the first page of the fibre seq of \( X \to \Sigma x \). \\

Exercise This generalizes the Borel construction: \( C_x(x \times E_{\Sigma x}) \) is a free resolution of \( C_x(x) \) over \( F_2[\Sigma x] \).

Let \( x^{\text{Fix}} \) be the fixed set of \( x \). If \( x \) has the homotopy type of a finite dim \( C_\Sigma \) space, there is a localization isomorphism:

\[
\Theta^{-1}H^*_B(x) \longrightarrow H^*(x^{\text{Fix}}, F_2) \otimes F_2[0, \theta^{-1}]
\]

3) a rank inequality

\[
\dim (H^*(x; F_2)) \geq \dim (H^*(x^{\text{Fix}}, F_2))
\]

(P. Smith 1938, but this formulation due to Borel)

Exercise This follows from considering the spectral sequence where we start with the box differentials. We crucially that if \( x \in x^{\text{Fix}} \), then \( 2 \theta \in x^{\text{Fix}} \).

So we have $H^*(x; F_2) \otimes F_2[\theta, 0^{-1}] \Rightarrow H^*(x^{\text{Fix}}, F_2) \otimes F_2[0, \theta^{-1}]$.

Note Everything but localization depends on the \( qi \) class of \( C_x \).

Note Finite-dimensionality (as an equivariant \( C_\Sigma \) space) is important:

\[
(x_1, x_2, \ldots) \Rightarrow (x_1 - x_2, x_3, \ldots)
\]
Exercise: Compute the spectral sequence for \( \cdots \quad \cdots \quad \cdots \)

\((x, x_1, x_2, x_3) \quad \downarrow \quad (x_0, x_1, x_2, x_3)\)

Let's try to do this in Lagrangian Floer.

- \((M, \omega, L)\) so symplectic involution, \(\tau^\omega = \omega\).
- \(\tau^2 = 1\) preserves \(L_0, L_1\) setwise.
- Fixed sets \(M^{\text{fix}} \cong M\), \(L_0^{\text{fix}} \leq L_0, L_1^{\text{fix}} \leq L_1\).

Exercise: If \(M^{\text{fix}}\) connected, show \(M^{\text{fix}}\) is symplectic and \(L_0^{\text{fix}}, L_1^{\text{fix}}\) are Lagrangians inside \(M\).

Q1. When/how can we define \(HF_{L_0}^{\omega} (L_0, L_1)\)?

Q2. When/how can we relate \(HF_{L_0}^{\omega} (L_0, L_1)\) to \(HF (L_0^{\text{fix}}, L_1^{\text{fix}})\)?

\(\text{want } CF (L_0, L_1) \circ \tau^\omega \text{ with } \tau^\omega \text{ a chain map.}\)

\[\text{w} \mapsto \text{Ext}_{\mathbb{Z}_2}^{\omega} (CF (L_0, L_1), \mathbb{Z}_2)\]

\(\text{and everything but localization follows.}\)

Plan: Find \(J(t)\) equivariant achieving transversality \(\cup^\omega\) a chain map.

Issue: This is hard! Equivariant \(\cup^\omega\) structures are not generic. Indeed, in some cases impossible.
Such a $J$ can always be found if there are no disks contained in the fixed set.

Tam (Seidel-Smith 2010) Under strict K-theoretic assumptions, $J$ an equivariant Hamiltonian isotopy of $L_0, L_1$ fixing $L_0^{\text{fix}}$ and $L_1^{\text{fix}}$ to $L_0', L_1'$ and $J(t)$ equivariant achieving transversality for $(L_0', L_1')$.

Furthermore $J$ a localization isomorphism $\theta_2^* \text{HF}_2(L_0', L_1') \to \text{HF}(L_0^{\text{fix}}, L_1^{\text{fix}}) \otimes \mathbb{Z}_2, \mathbb{L}, e^{-J}$.

What assumptions? Recall the problem had something to do with relative K-theory

$$\begin{align*}
N_0^{\text{fix}} \times S^0 & \cong N_0^{\text{fix}} \times [0, 1] \quad \text{and} \\
N_1^{\text{fix}} \times S^0 & \cong N_1^{\text{fix}} \times [0, 1] \\
L_0 \times S^0 & \cong M^{\text{fix}} \times [0, 1] \\
L_1 \times S^0 & \cong M^{\text{fix}} \times [0, 1] \\
\end{align*}$$

Want relative triviality (as a cpx bundle)

$$(N_0^{\text{fix}} \times [0, 1], (N_0^{\text{fix}} \times S^0) \otimes \mathbb{L}) \cong (N_1^{\text{fix}} \times [0, 1], (N_1^{\text{fix}} \times S^0) \otimes \mathbb{L})$$
Lots of spectral sequences

<table>
<thead>
<tr>
<th>Seidel</th>
<th>Smith</th>
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</thead>
<tbody>
<tr>
<td>$K_h \text{symp} (K)$</td>
<td>$\widehat{HF} (\Sigma (K))$</td>
</tr>
<tr>
<td>$K_h \text{symp} (\wedge K)$</td>
<td>$K_h \text{symp} (K)$</td>
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<tr>
<td>$\widehat{HF}_K (\Sigma (K), \wedge K)$</td>
<td>$\widehat{HF}_K (S^3, K)$</td>
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<tr>
<td>$HF (\psi^2)$</td>
<td>$HF (\psi)$</td>
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</tbody>
</table>

Reproved by Seidel in a much broader setting.

Plan 2: Lots of nonequivalent complex structures

Version 1 (Seidel and Smith) Heavily modeled on Morse theory, more or less an $\infty$-version worth of $J$'s.


Version 3 (Cao-Honda) Kuranishi structures, fairly different.
Construction \((M, l_0, l_1, m) \sim \tilde{CF}(l_0, l_1)\), the Freed Floer complex

\[ \tilde{CF}(l_0, l_1) \cong CF(l_0, l_1) \text{ as } \mathbb{F}_2\text{-complexes; } \tilde{CF}(l_0, l_1) \text{ is a Free } \mathbb{F}_2[\mathbb{Z}_2]\text{-complex.} \]

**Thm 1** A well-defined \(HF_{\mathbb{Z}_2}(l_0, l_1)\) which is an invariant of the equivariant Hamiltonian isotopy type of \(l_0 \equiv l_1\), and a spectral sequence

\[ HF(l_0, l_1) \otimes \mathbb{F}_2[\mathbb{Z}_2] \Rightarrow HF_{\mathbb{Z}_2}(l_0, l_1) \] each page of which is similarly an invariant.

**Thm 2** IF \(F\) an equivariant configuration achieving transversality (e.g. if the S-S conditions are satisfied, or we have a nice diagram) then \((CF(l_0, l_1), \Sigma) \cong \tilde{CF}(l_0, l_1)\) as \(\mathbb{F}_2[\mathbb{Z}_2]\)-complexes. (So every page of the spectral sequences in the table is an inv of the equivariant Hamiltonian isotopy type of \(l_0 \equiv l_1\), therefore generally the topological input data).
Let $K$ be a knot and $\Sigma(K)$ its double. There is a well-defined $\mathbb{F}_2[\ell]$-module $\overline{\mathrm{HF}}_{\mathbb{F}_2}(\Sigma(K))$ with the property that $\theta^{-1}\overline{\mathrm{HF}}_{\mathbb{F}_2}(\Sigma(K)) \cong \mathbb{F}_2[\ell, \theta^{-1}]$.

Get a spectral sequence $\overline{\mathrm{HF}}(\Sigma(K)) \otimes \mathbb{F}_2[\ell] \Rightarrow \overline{\mathrm{HF}}_{\mathbb{F}_2}(\Sigma(K))$.

**Proof.** Let $\sigma(K)$ be twice the minimal homological grading in $\overline{\mathrm{HF}}_{\mathbb{F}_2}(\Sigma(K))$.

**Proposition (HLS).** $\sigma$ is a concordance inv.

**Examples.**

- $\Sigma(2, 3, 7) = \Sigma(T(3, 7)) = \Sigma(P(-2, 3, 7))$
- $\overline{\mathrm{HF}}(\Sigma(2, 3, 7)) = \overline{\mathrm{HF}}(\Sigma(T(3, 7))) = \overline{\mathrm{HF}}(\Sigma(P(-2, 3, 7)))$

Involution is trivial for $T(3, 7)$

Nontrivial for $P(-2, 3, 7)$

**Exercise.** This determines the spectral sequence, and implies

- $\sigma(T(3, 7)) = 0$
- $\sigma(P(-2, 3, 7)) = -2$
Similarly for $HF^-$:

$$
\begin{pmatrix}
0 & -2 & -2 \\
-2 & & \\
-4 & & \\
\end{pmatrix}
$$

The $(HLS)$ is an equivariant $HF_{Z_2}(\Sigma(K), s_0)$, which is an $\mathbb{F}_2[\theta]$-module.

There is a spectral sequence with $E_1 \cong HF^-(\Sigma(K), s_0) \otimes \mathbb{F}_2[\theta]$.

$\theta$-equivariance: Arrows go from tail to $\theta$-torsion elements.

\[ d_\infty(K, 1) \leq d_\infty(K, 2) \leq \frac{1}{2} \sum (\Sigma(K), s_0) \]

Concordance inuts, not homomorphisms.

Exercise: What are $d_\infty(K, 2)$ of $T(3, 7)$, $P(-2, 3, 7)$, and their mirrors?