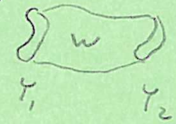
Lecture 23

Intro to Involutive

Context/Background

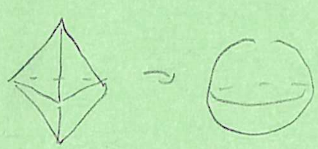
Recall we have $\Theta^3_{\mathbb{Z}} : \left(\{ Y_i \text{ oriented } \mathbb{Z}HS^3 \}, \# \right) \Big/$ smooth



$H_*(Y_i) \xrightarrow{\sim} H_*(W$

Freedman Every $\mathbb{Z}HS^3$ bounds an acyclic topological ball. So this is about the smooth-vs-topological categories.

Defn A triangulation of a manifold Y is a homeomorphism $\varphi : |X| \to Y$, where $|X|$ is the geometric realization of a simplicial complex X.



Cairns, Whitehead Any smooth manifold admits a (smooth) triangulation (this means the restriction to any open simplex is a smooth embedding).

- For dimensions ≤ 3 everything has a unique smooth structure (Rado, Moise)

Casson ('90) Freedman's E_8 mfd is a nontriangulable 4-mfd.

Galewski-Stern, Matumoto '70s $\exists$ a nontriangulable mfd in every dimn $\geq 5 \iff \exists$ an element of order 2 in $\Theta^3_{\mathbb{Z}}$ w/ Rokhlin invt one.

Recall the Rokhlin homomorphism is

$$\mu : \Theta^3_{\mathbb{Z}} \twoheadrightarrow \mathbb{Z}/2, \quad \mu(Y) = \frac{\sigma(W)}{8} \pmod 2 \quad \text{w/}$$

W any cpt smooth 4-mfld w/ bdy Y.

__Example__ $\Sigma(2,3,5)$ bounds $-E_8$ which has signature -8, so $\mu(\Sigma(2,3,5)) =$

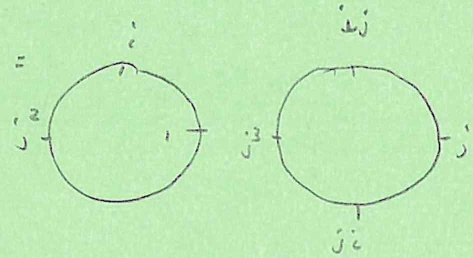
The condition above is the same thing as

$$0 \longrightarrow \ker(\mu) \longrightarrow \Theta^3_{\mathbb{Z}} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0 \quad \text{does not split}$$

__Manolescu, 2013__ Said exact sequence does not split.

__How?__ Equivariant version of Seiberg-Witten Floer homology

- SWFH is the homology of a spectrum X constructed from solutions to the Seiberg-Witten equations on Y.
- Come w/ an action of $Pin(2)$:



$j^2 = -1$
$ij = -ji$

[Side note: $0 \to \mathbb{Z}_2 \to Pin(2) \to O(2) \to 0$]

- $H_*^{Pin(2)}(X)$ is a module over $H^*(BPin(2); \mathbb{F}_2) \simeq \mathbb{F}_2[u,v]/_{u^3=0}$ $\quad \deg(u) = 1$, $\deg(v) = 4$
- Three correction term style invariants: $\alpha, \beta, \gamma : \Theta^3_{\mathbb{Z}} \to 2\mathbb{Z}$, __not__ homomorphisms.
- $\beta(Y) = -\beta(-Y)$ and $\beta(Y) \equiv \mu(Y) \bmod (2)$. So can't have $[Y] = [-Y]$ __and__ $\mu(Y) = 1$

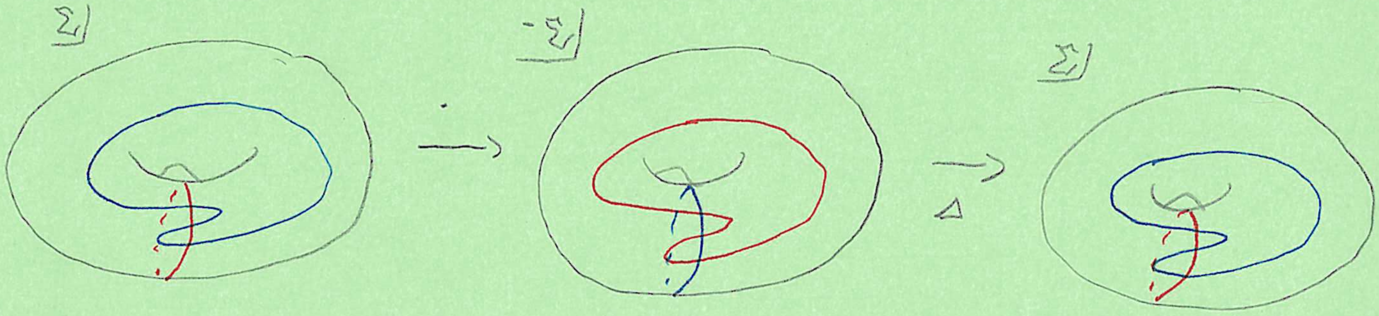
Awesome! But generally hard to compute.

Heegaard Floer

- Lidman-Manolescu, Kutluhan-Lee-Taubes, Colin-Ghiggini-Honda + Taubes:

  $HF^+(Y) \simeq S^1$-equivariant SWFH

- wrt that equivalence, we can construct the analog of the "involution" j_*.

- 

$$\begin{tikzcd} CF^\circ(\mathcal{H},s) \arrow[r, mapsto, "\eta"] \arrow[rr, bend right, "\iota"'] & CF^\circ(\bar{\mathcal{H}},\bar{s}) \arrow[r, "\Phi(\bar{\mathcal{H}},\mathcal{H})"] & CF^\circ(\mathcal{H},\bar{s}) \end{tikzcd}$$

- $\iota^2 \sim Id$ (uses Juhasz-Thurston naturality for HF).
- Ideally one would want to construct $\mathbb{Z}_2$-equivariant (co)homology; this would give an overall $H^*(BPin(2);\mathbb{F}_2)$-module structure.
- However

$$\begin{tikzcd} CF^+(Y) \arrow[r, "1+\iota"] \arrow[rr, bend right] \arrow[rrr, bend right] & QCF^+(Y) \arrow[r, "1+\iota"] \arrow[rr, bend right] \arrow[rrr, bend right] & Q^2CF^+(Y) \arrow[r, "1+\iota"] \arrow[rr, bend right] & Q^3CF^+(Y) \arrow[r] & \cdots \end{tikzcd}$$

- Don't know how to define H_2 (or even show H_1 ind of choices).

So instead $\quad CFI^+(Y,s) \simeq \mathrm{Cone}\left(CF^+(Y) \xrightarrow{1+\iota} QCF^+(Y)\right)$

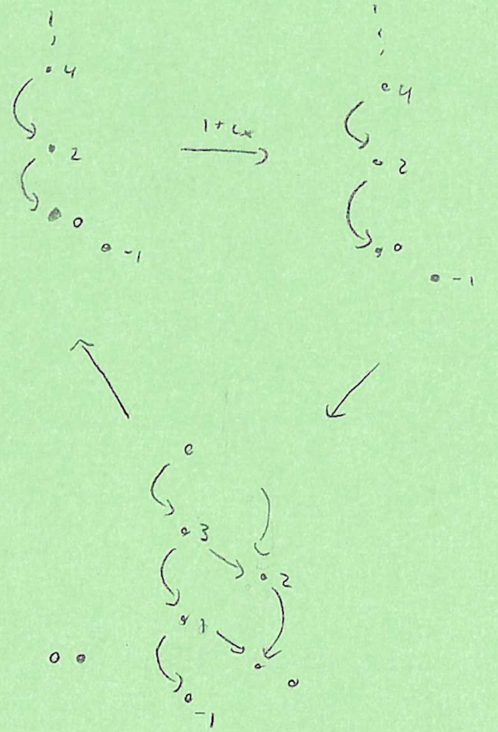
- Module over $\mathbb{F}[U,Q]/Q^2 \cong H^*(B\mathbb{Z}_4;\mathbb{F}_2)$. Notionally corresponds to $\mathbb{Z}_4$-SWFH.
  $\deg(Q)=1$

- What kind on new invariants do we get?

Automatically have an exact triangle:

$$\begin{tikzcd} HF^\circ(Y,s) \arrow[rr, "1+\iota_*"] & & HF^\circ(Y,s) \arrow[dl] \\ & HFI^\circ(Y,s) \arrow[ul] & \end{tikzcd}$$

Example $\Sigma(2,3,7)$



For $s=\bar{s}$,

We get two new correction terms:

$\bar{d}(Y,s) = \{$Minimal grading in right hand tower; $U^n Q\, HFI^+(Y,s)$, $n \gg 0\}$

$\underline{d}(Y,s) = \{$Minimal grading in lefthand tower; odd degrees of $U^n HFI^+(Y,s)$, $n \gg 0\} - 1$

eg $\underline{d}(\Sigma(2,3,7)) = -2, \quad \bar{d}(\Sigma(2,3,7)) = 0 = d(\Sigma(2,3,7))$

Q What are they for L-spaces?

Note How do we show this is an invt?

$$\begin{tikzcd}
CF^\circ(H_1,s) \arrow[r, "1+\iota_1"] \arrow[d, "\Phi(H_1,H_2)"'] \arrow[dr, "\Psi"] & CF^\circ(H_1,s) \arrow[d, "\Phi(H_1,H_2)"] \\
CF^\circ(H_2,s) \arrow[r, "1+\iota_2"] & CF^\circ(H_2,s)
\end{tikzcd}$$

Ψ is a homotopy between $\iota_2 \circ \Phi$ and $\Phi \circ \iota_1$.

$$\begin{aligned}
CFI^\circ(H,s) &\longrightarrow CFI^\circ(H_2,s) \\
(a,b) &\longmapsto (\Phi(a), \Psi(a) + \Phi(b))
\end{aligned}$$

$$\begin{tikzcd}
(a,b) \arrow[r] \arrow[d] & (\Phi(a), \Psi(a) + \Phi(b)) \arrow[d] \\
(\partial a, a + \iota_1 a + \partial b) \arrow[r] & (\partial\Phi a, \underline{\iota_2\Phi a} + \Phi a + \underline{\partial\Psi(a)} + \partial\Phi(b))
\end{tikzcd}$$

$$(\Phi(\partial a), \underline{\Psi(\partial a)} + \Phi(a) + \underline{\Phi\iota_1(a)} + \Phi(\partial b))$$

One also gets concordance invts in the usual way

$$\underline{V}_0 = \frac{-1}{2}\left(\underline{d}(S^3_n(K), [0])\right) - \frac{n-1}{4}$$

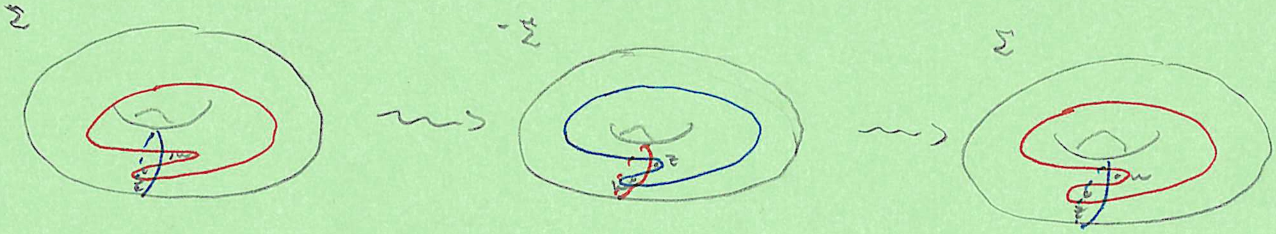
$$\bar{V}_0 = \frac{-1}{2}\left(\bar{d}(S^3_n(K), [0])\right) - \frac{n-1}{4}$$

How would you compute such a thing?

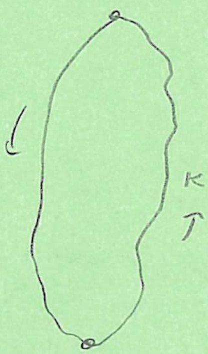
First method: Via (large) surgery.

Could rerun original algorithm w/ a knot diagram:

Σ $\quad$ $-\Sigma$ $\quad$ Σ

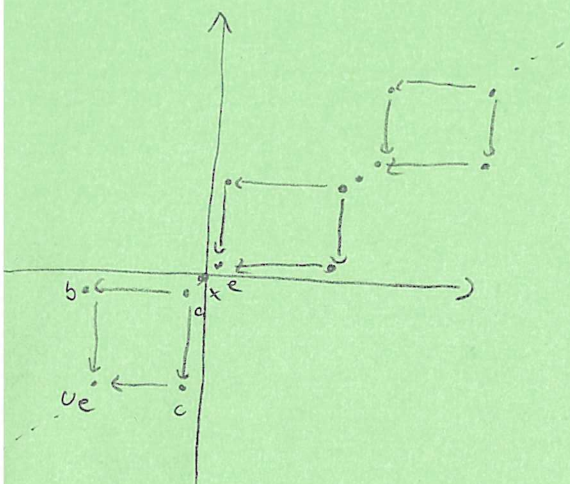


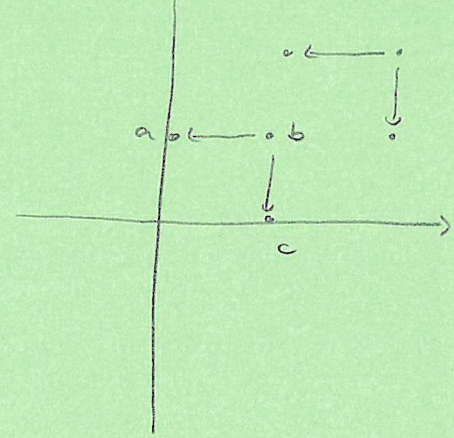
The corresponding map ι_K squares to the so-called Sarkar involution



corresponding to moving the basepoint around the knot (this is the map that pushes in a circle around the core circle of a torus and leaves the boundary fixed).

On CFK^∞, this map is $Id + U^{-1}\Phi\Psi$, where Ψ is odd length horizontal and Φ is odd-length vertical.





$\sigma|$ $a \mapsto a+e$
$b \circlearrowleft$ $c \circlearrowleft$ $e \circlearrowleft$ $x \circlearrowleft$

$\sigma = Id$

Exercise

Up to change of basis, this is the only

$\iota_K|$ $a \mapsto a+x$
$x \mapsto x+e$
$b \mapsto c$ $\quad$ $e \circlearrowleft$

$\iota_K|$ $a \leftrightarrow b$
$c \circlearrowleft$

Large Surgery Formula

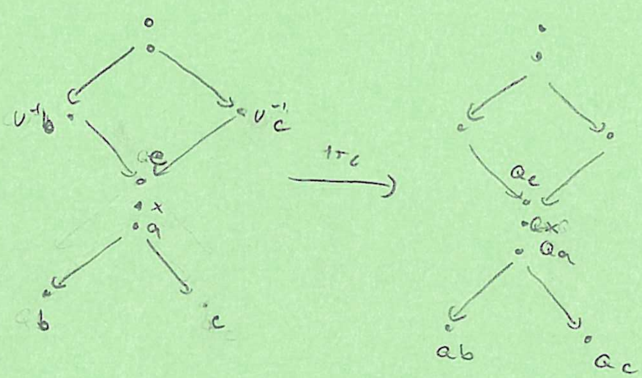
For $n \geq g(K)$, $CFI^+(S^3_n(K)) = \mathrm{Cone}\left(A_0^+ \xrightarrow{1+\iota_K} A_0^+\right)$

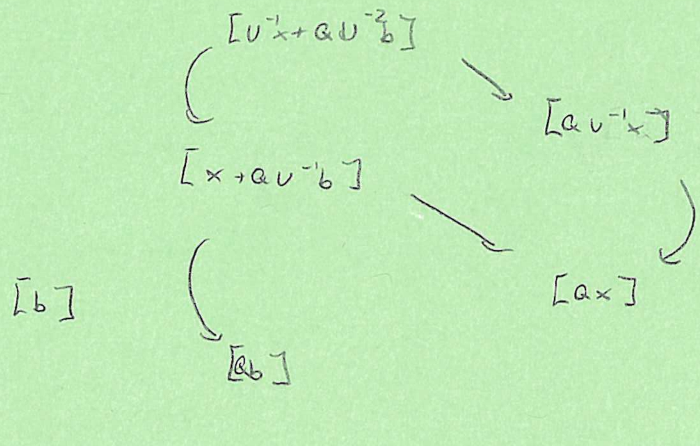
Why would we not have a general surgery formula? There is a problem w/ cobordism triples.

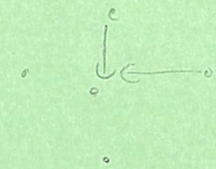
$$(\alpha, \beta, \gamma) \rightsquigarrow (\beta, \alpha, ?)$$

[There is a surgery triangle for the hat version using the bordered construction; perhaps more on this later.]

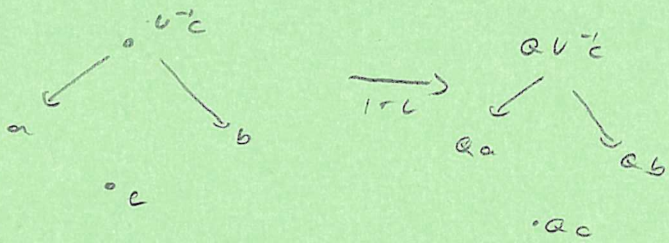
eg $\Sigma(2,3,7)$





Could also have done this w/ 

$\Sigma(2,3,7)$



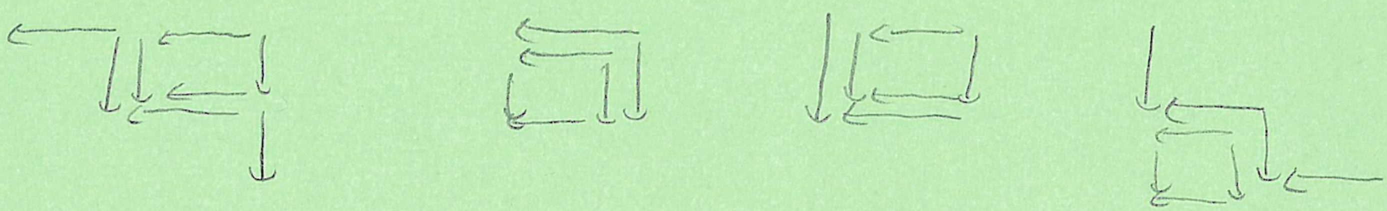
From observing that you should be able to dualize a complex for $S^3_{+1}(LHT)$.

Note that this means we can compute $\underline{V}_0$, $\overline{V}_0$ just by comparing the lengths of the towers in $H_*(AI_0^+)$ to the length of the tower in $H_*(A_0^+)$.

Exercise What is $\underline{V}_0$, $\overline{V}_0$ of your favorite L-space knot?

Exercise What is $\underline{V}_0$, $\overline{V}_0$ of the possible arrangements of thin knots?

[There are a large number of separate cases.]



How does any of this behave wrt connected sum?

For that we need to move to CF^-...