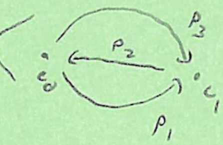
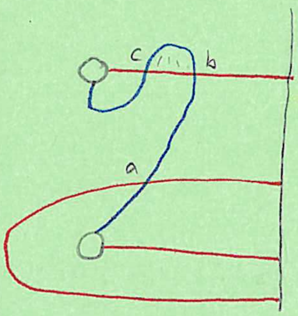
Lecture 20

Last Time $A(\mathcal{Z})$, $\widehat{CFA}$, $\widehat{CFD}$

eg $A(T^2) = \mathbb{F}\Big\langle$  $\Big\rangle \Big/ \rho_2\rho_1 = \rho_3\rho_2 = 0$

Eight generators

$\widehat{CFA}$: Counts hol'c curves w/ asymptotics on $\mathcal{Z}$ as a multiplication

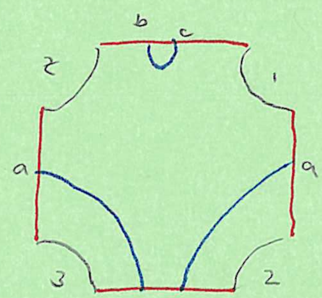


$m_1(c) = b$

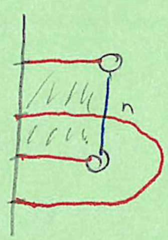
$m_2(c, \rho_2) = a$

$m_2(a, \rho_3) = b$

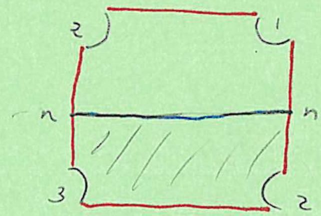
$m_2(c, \rho_{23}) = b$



$\widehat{CFD}$: Counts hol'c curves w/ asymptotics on $\mathcal{Z}$ as a differential.



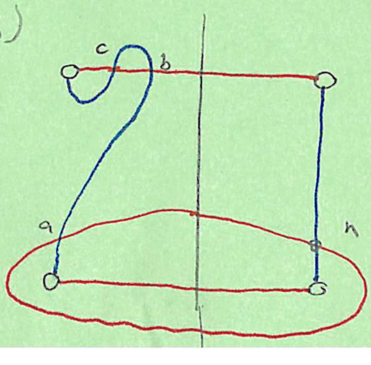
$\partial_1(n) = \rho_{23} n$

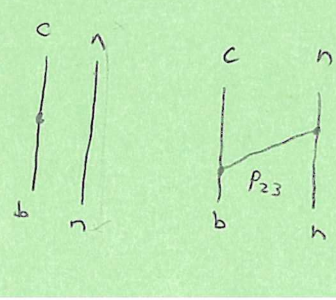


$\widehat{CFA}(\mathcal{H}_0) \boxtimes \widehat{CFD}(\mathcal{H}_1)$

$\uparrow$

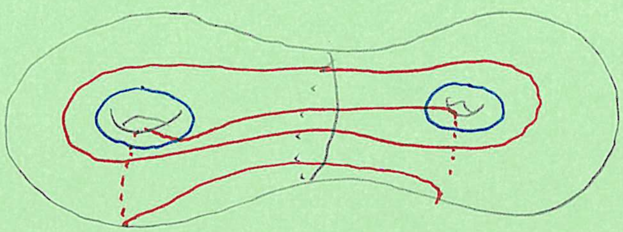
$\mathbb{F}\langle \iota_0, \iota_1 \rangle$





$\partial(c \boxtimes n) = 2(b \boxtimes n)$
$= 0$

Note

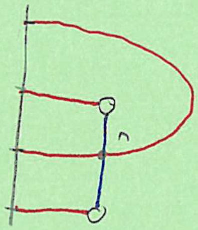


$$H_1(Y) = \langle \lambda_1, \lambda_2, \mu_1, \mu_2 \rangle \Big/ \lambda_1, \lambda_2, \lambda_1+\lambda_2, \mu_1+\mu_2$$

$$\cong \mathbb{Z}$$

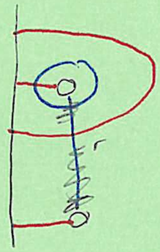
Surgery exact triangle from

$\mathcal{H}_0$



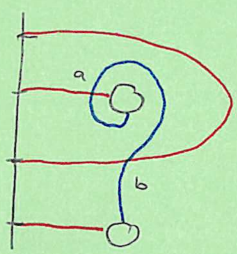
$\partial(n) = \rho_{12}\, n$

$\mathcal{H}_\infty$



$\partial(r) = \rho_{23}\, r$

$\mathcal{H}_{-1}$



$\partial a = \rho_3 b + \rho_1 b$

$$0 \longrightarrow \widehat{CFD}(\mathcal{H}_\infty) \xrightarrow{\phi} \widehat{CFD}(\mathcal{H}_{-1}) \xrightarrow{\psi} \widehat{CFD}(\mathcal{H}_0) \longrightarrow 0$$

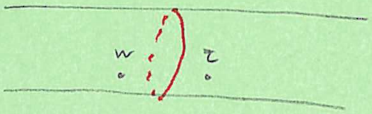
$$r \longmapsto b + \rho_2 a$$

$$a \longmapsto n$$

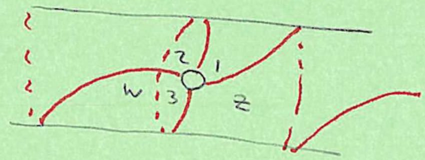
$$b \longmapsto \rho_2 n$$

Let $Y = S^3 - nbd(K)$ w/ some framing n. There is an algorithm $CFK^\infty(K) \rightsquigarrow \widehat{CFD}(Y)$.

How this algorithm essentially works: Take α to be a meridian for your knot.

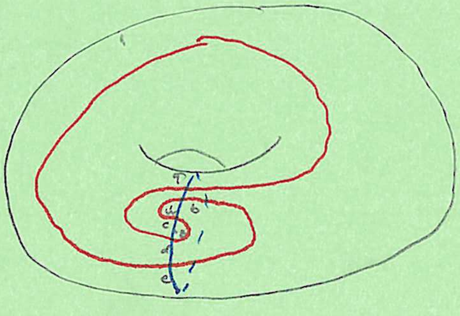


$\rightsquigarrow$ Puncture here and add an α curve for one longitude



Let $CFK^\infty(K)$ have a basis that is both vertically and horizontally simplified. [There is a way to do this if you don't have that, via keeping track of a change of basis matrix.

eg]



$$\begin{tikzcd} & & e \arrow[dll, bend right] \arrow[ddl, bend left] \\ d & a \arrow[l] \arrow[d] & \\ c & b \arrow[l] & \end{tikzcd} \otimes \mathbb{Z}[U, U^{-1}]$$

$\rightsquigarrow$ replace w/ a vertically simplified basis

$$\begin{tikzcd} d \arrow[d] & a \arrow[l] \arrow[d] & e \\ c & b \arrow[l] & \end{tikzcd}$$

From CFK^∞ to $\widehat{CFD}(Y)$:

- $\iota_0 \widehat{CFD}(Y)$ has one generator for each basis element x_i.
- $\iota_1 \widehat{CFD}(Y)$ has elements coming from the arrows, plus some additional elements
- To a vertical arrow of length ℓ $\begin{array}{c} x_1 \\ \downarrow \ell \\ x_2 \end{array}$ we associate

$$\begin{tikzcd}
x_1 \arrow[d, "\rho_1"] \\
\kappa_1 \\
\cdots \arrow[u, "\rho_{23}"] \\
\kappa_\ell \arrow[u, "\rho_{23}"] \\
x_2 \arrow[u, "\rho_{123}"]
\end{tikzcd}$$

- To a horizontal arrow of length ℓ $x_2' \xleftarrow{\ell} x_1'$ we associate the chain

$$x_2' \xleftarrow{\rho_2} \lambda_\ell' \xleftarrow{\rho_{23}} \cdots \xleftarrow{\rho_{23}} \lambda_1' \xleftarrow{\rho_3} x_1'.$$

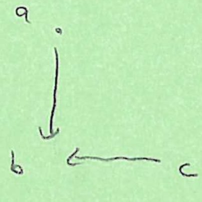
Finally between x_v and x_h the distinguished horizontal and vertical arrows we have the <u>unstable chain</u> which varies based on framing:

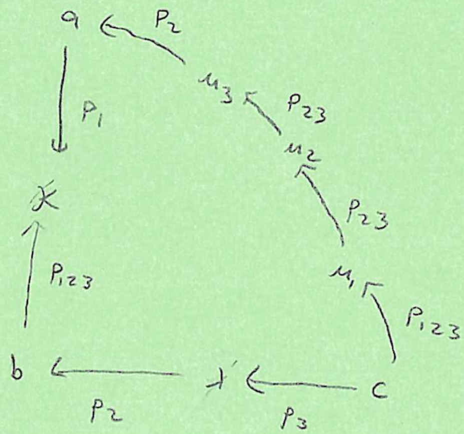
- When $n < 2\tau(K)$, $x_v \xrightarrow{\rho_1} z_1 \xleftarrow{\rho_{23}} z_2 \longleftarrow \cdots \xleftarrow{\rho_{23}} z_m \xleftarrow{\rho_3} x_h \qquad m = 2\tau(K) - n)$
- When $n = 2\tau(K)$, $x_v \xrightarrow{\rho_{12}} x_h$
- When $n > 2\tau(K)$, $x_v \xrightarrow{\rho_{123}} \lambda_1 \xrightarrow{\rho_{23}} \cdots \xrightarrow{\rho_{23}} \lambda_m \xrightarrow{\rho_2} x_h \qquad m = |2\tau(K) - n|$

Examples

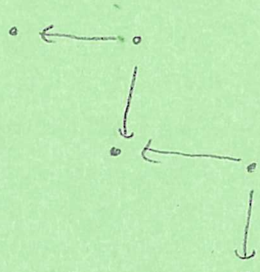
• LHT

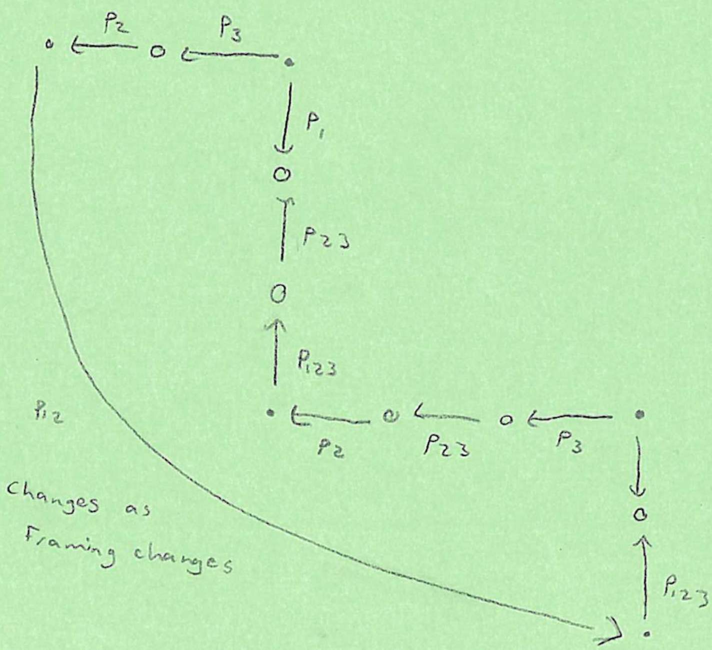
Framing 1



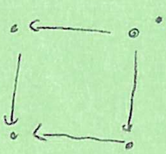


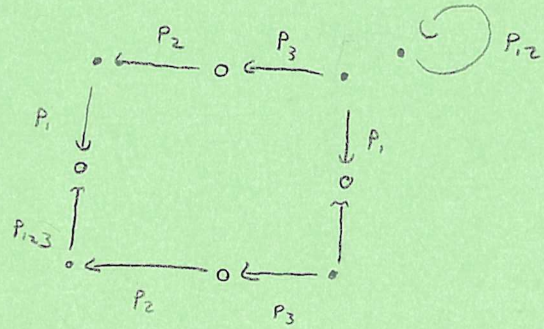
• $T_{3,4}$ Framing $2\tau=4$





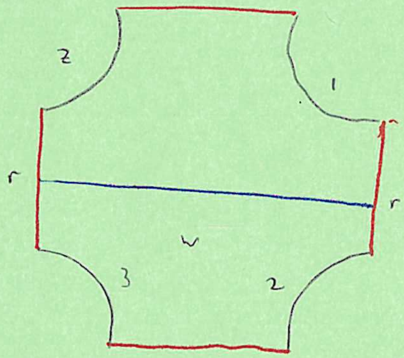
• 4_1 0-Framing





Now I can do various things w/ this knot complement.

Example 0 Glue the knot back in



$m_{3+i}(r, \rho_3, \rho_{23}, \ldots, \rho_{23}, \rho_2) = a$

Note $\iota_0 r$

eg $T_{3,4}$, 0-Framing

$$\begin{tikzcd}
x_1 \arrow[ddddrrrrr, bend right, "\rho_{12}"'] & o \arrow[l, "\rho_2"'] & x_2 \arrow[l, "\rho_3"'] \arrow[d, "\rho_1"] & & & \\
& & o & & & \\
& & o \arrow[u, "\rho_{23}"'] & & & \\
& & x_3 \arrow[u, "\rho_{23}"'] & o \arrow[l, "\rho_2"'] & o \arrow[l, "\rho_{23}"'] & x_4 \arrow[l, "\rho_3"'] \arrow[d, "\rho_1"] \\
& & & & & o \\
& & & & & x_5 \arrow[u, "\rho_{123}"']
\end{tikzcd}$$

Generators: $r x_i$ } Five of them.

Nothing else is in ι_1

$\partial(rx_1) = 0$

$\partial(rx_2) = rx_1$

$\partial(rx_3) = 0$

$\partial(rx_4) = rx_3$

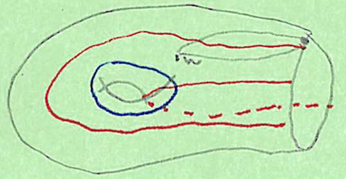
$\partial(rx_5) = 0$

$\rightsquigarrow \widehat{HF}(S^3) = \langle [rx_5] \rangle$

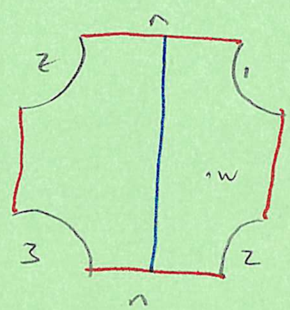
How do I instead recover CFK^-? I add a basepoint, and work over $\mathbb{F}[U]$.

$\partial(rx_2) = U r x_1$

$\partial(rx_4) = U^2 r x_3$



Example 2   0-surgery on 4_1



$$m_{3+i}(n, \rho_2, \rho_{12}, \ldots, \rho_{12}, \rho_1) = n$$

$$\begin{tikzcd}
\bullet \arrow[d, "\rho_1"'] & x_1 \arrow[l, "\rho_2"'] & \bullet \arrow[l, "\rho_3"'] \arrow[d, "\rho_1"] \\
x_2 & & x_4 \\
\bullet \arrow[u, "\rho_{123}"] & x_3 \arrow[l, "\rho_2"] & \bullet \arrow[l, "\rho_3"] \arrow[u, "\rho_{123}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[loop right, "\rho_{12}"]
\end{tikzcd}$$

$\partial(n_{x_1}) = b_{x_2}$

$\partial(n_{x_2}) = 0$

$\partial(n_{x_3}) = 0$

$\partial(n_{x_4}) = 0$

So $rk(\widehat{HF}(S^3_0(4_1)) = 2.$

$rk(\widehat{HF}(S^3_0(4_1), \hat{K}) = 4.$

We can also use this to study satellite operators (for example cabling).

Consider P a pattern knot inside a solid torus $D^2 \times S^1$.

We can look at $\widehat{CFA}(D^2 \times S^1, P)$.

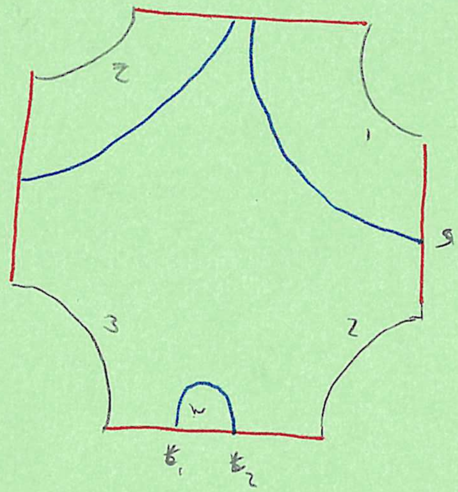
First some explanation of doubly-pointed diagrams:

Say I have $\mathcal{H}_1$ is a bordered Heegaard diagram for $Y - nbd(K)$, and $\mathcal{H}_0$ is a bordered diagram representing $D^2 \times S^1$ w/ the ∞-framing. If I put an additional basepoint w in $\mathcal{H}_0$, then

$\mathcal{H}_0 \bigsqcup_{T^2} \mathcal{H}_1$ is a doubly-pointed Heegaard diagram for $S^3, P(K)$, where

P is <u>some</u> pattern knot.

I can put in a pattern for eg $T_{2,1}$

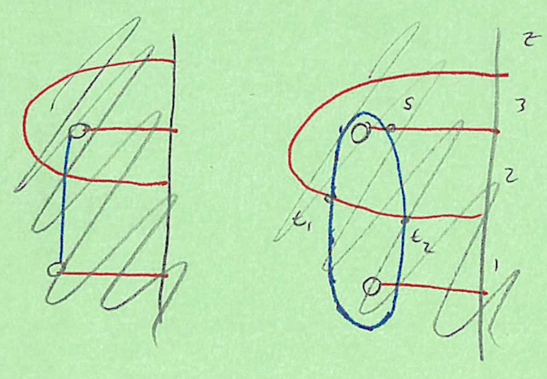


$m_2(s, \rho_1) = t_2$

$\widehat{CFA}(T_{2,1}) \boxtimes \widehat{CFD}(Y)$ $\quad Y - S^3 - nbd\, K$ n-framed

$\Rightarrow \widehat{HFK}(K_{2,2n+1})$

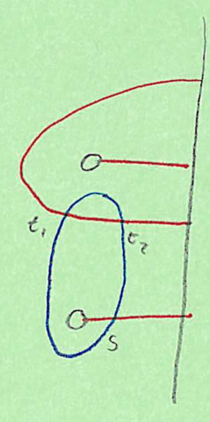
<u>Ok so</u> $\widehat{CFA}(T_{2,1}) \otimes \widehat{CFD}(LHT)$



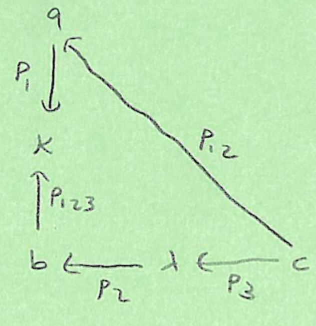
$s\,\iota_0$

$t_1\,\iota_1$

$t_2\,\iota_1$



<u>2-Framed LHT complement</u> Generators sa, sb, sc, $t_1\kappa$, $t_2\kappa$, $t_1\lambda$, $t_2\lambda$



If I just want $\widehat{CFK}$:

$\partial(sa) = (t_2\,\kappa)$

IF I want CFK^- $\quad m_2(s, p_1) = t_2 \qquad m_2(s, p_{23}) = U^2 s$

$m_1(t_1) = U t_2 \qquad m_2(s, p_{123}) = U t_2$

$\rightsquigarrow$ many more things

$$\begin{tikzcd} sb \arrow[dr, "\cdot U"'] & sa \arrow[d] & t_1 x \arrow[dl, "\cdot U"] \\ & t_2 x & \end{tikzcd} \qquad \begin{tikzcd} t_1 \lambda \arrow[d, "\cdot U"] \\ t_2 \lambda \end{tikzcd}$$

sc

$\mathbb{F}[U]\,[Usa + t_1 x] \qquad [t_2 \lambda] \qquad [sb + Usa]$

$[Usb + U^2 sa]$

<u>More generally</u>

Say that I have $\psi : F(\mathcal{Z}_1) \xrightarrow{\sim} F(\mathcal{Z}_2)$. Then we can consider the mapping cylinder $M_\psi = ([0,1] \times F(\mathcal{Z}_2),\ \psi : F(\mathcal{Z}_1) \to \{0\} \times F(\mathcal{Z}_2),$

$id : F(\mathcal{Z}_2) \to \{1\} \times F(\mathcal{Z}_2))$

We can associate bimodules to M_ψ, and thus to the mapping class: $\widehat{CFDD}(M_\psi)$, $\widehat{CFAA}(M_\psi)$, $\widehat{CFDA}(M_\psi)$.

<u>Why should you care?</u>

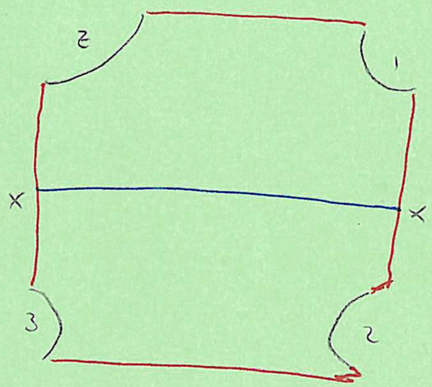
- Given Y a 3-mfd, can split Y as two handlebodies glued along a diffeomorphism $H_1 \cup_\varphi H_2$, $\varphi : F \to F$.

- Can split Y as two standard bordered handlebodies plus a diffeomorphism $\psi : F(\mathcal{Z}_1) \to F(\mathcal{Z}_2)$

- $\widehat{CF}(Y) \simeq \widehat{CFA}(H_0) \boxtimes \widehat{CFDA}(\psi) \boxtimes \widehat{CFD}(H_0)$

- <u>Moreover</u>
  - This only depends on the isotopy class of ψ
  - This factors as $\widehat{CFDA}(\psi \circ \varphi) = \widehat{CFDA}(\psi) \boxtimes \widehat{CFDA}(\varphi)$.
  - So it suffices to compute $\widehat{CFDA}$ for some set of generators of the mapping class group of the torus
  - It's actually better than this; we can use morphism spaces to only actually need CFDD.

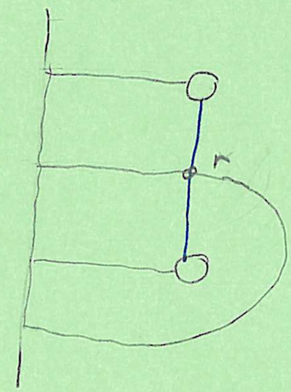
If I have $\widehat{CFD}(Y-U)$, I can produce $\widehat{CFA}$



$x \circlearrowleft \rho_{23}$

$\wr$

$x \circlearrowleft \rho_{12}$



$\widehat{CFAA}(\mathbb{I}) \boxtimes \widehat{CFD}(Y)$

<u>Generators</u> $\vec{z}_1 r$ $\quad \vec{z}_2 r$ $\quad \vec{x} r$

$$\begin{tikzcd} & \vec{z}_1\vec{r} \arrow[dl, "\sigma_2"'] \arrow[dr, "1+\sigma_{23}"] & \\ \vec{x}r \arrow[rr, "\sigma_3"'] & & \vec{z}_2\vec{r} \end{tikzcd}$$

S1


$(\sigma_3, \sigma_2) + (\sigma_3; \sigma_{23}, \sigma_2)$

$+ \dots$

<u>Example</u>

Hedden-Levine: Let Y_1 and Y_2 be L-space homology spheres, and $K_1 \subseteq Y_1$, $K_2 \subseteq Y_2$ nontrivial knots, Then the 0-framed splicing of the complement is not an L-space

Hanselman: conditions under which splicing any integer-framed complements returns an L-space

↖Also a calculus for torus boundary mfds seeing $\widehat{CFD}$ in terms of intersections of immersed curves.