Lecture 2

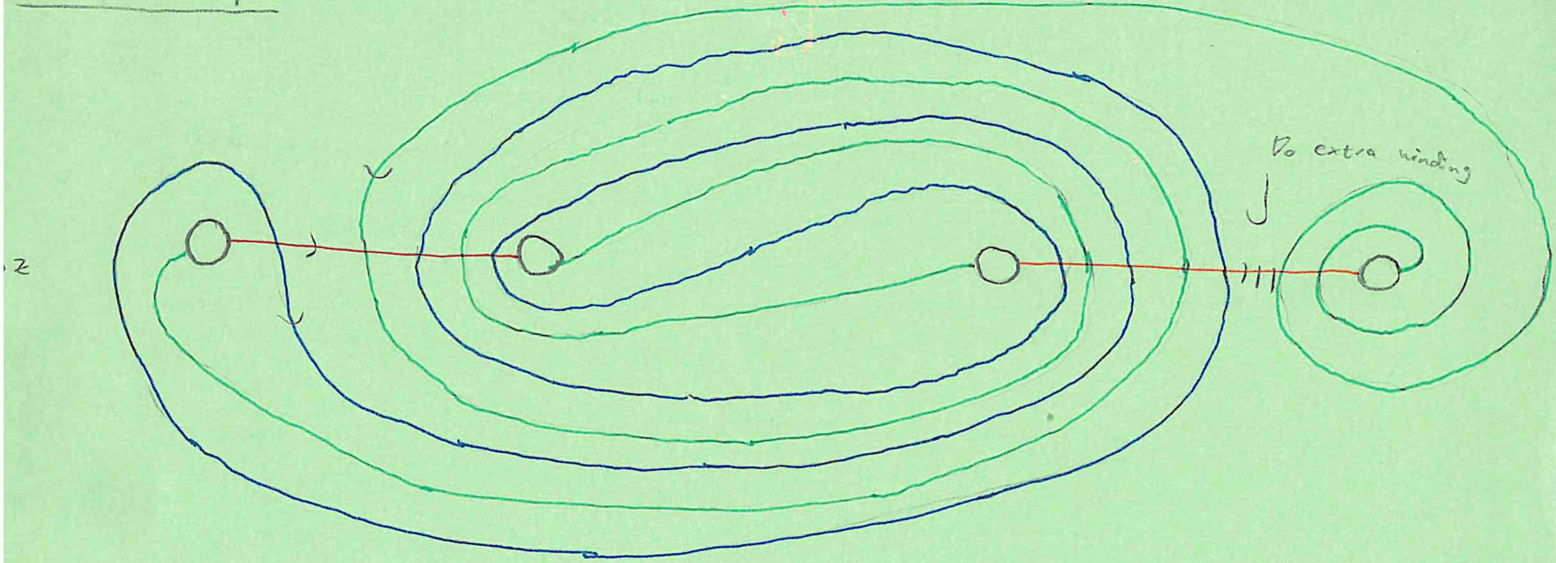
Last Time A (singly-pointed) Heegaard diagram consists of $\mathcal{H} = (\Sigma, \vec{\alpha}, \vec{\beta}, z)$ w/

- Σ a surface of genus g
- $\vec{\alpha} = (\alpha_1, \ldots, \alpha_g)$ (respectively $\vec{\beta} = (\beta_1, \ldots, \beta_g)$) are a collection of nonintersecting simple closed curves of Σ st $g = \dim \operatorname{Span} \{[\alpha_i]\} \subseteq H_1(\Sigma; \mathbb{R})$, and $\alpha_i \pitchfork \beta_j$.
- z is a basepoint in $\Sigma - \{\vec{\alpha}\} - \{\vec{\beta}\}$.

A Heegaard diagram gives us a unique way of constructing a 3-mfd

Last Time Some genus one Heegaard diagrams.

Another example



Exercise $H_1(Y) = \mathbb{Z}_2$

w/ one or three extra wrappings,

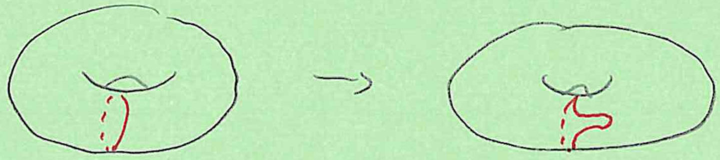
$H_1(Y) = \phi$.

$\Sigma(2,3,5) / \Sigma(2,3,7)$

When do two Heegaard diagrams give the same 3-mfd?

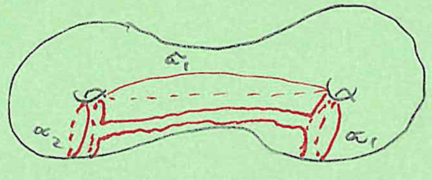
When they are connected by <u>Heegaard moves</u>.

① Isotopy of curves not crossing the basepoint.



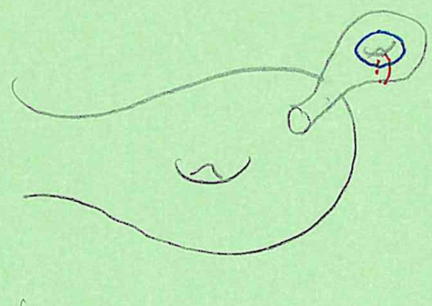
② Handleslides amongst α (respectively β) curves

Preserves the Heegaard decomposition



$\alpha_1, \alpha_2, \tilde{\alpha}_1$ bound an embedded pair of pants in $\Sigma - \{\alpha_3, \ldots, \alpha_g\}$

③ (De)<u>Stabilization</u>



Connect sum w/ a genus 1 surface as shown.

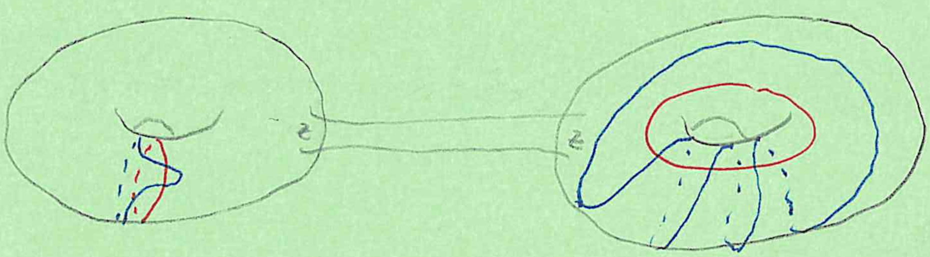
Note that this has the effect of connect summing your mfd w/ S^3.

<u>Thm</u> (Singer) Given two Heegaard decompositions (Y, U_0, U_1) and (Y, U_0', U_1') of a 3-mfd Y having genus g and g' respectively, $\exists$ g st the $(k-g)$-fold stabilization of the first decomposition is diffeomorphic to the $(k-g')$-fold stabilization of the second decomposition

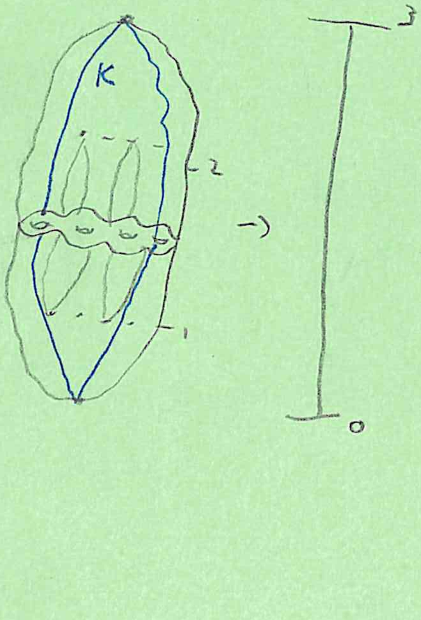
Thm Let U be a handlebody of genus g, and let $(\alpha_1, \dots, \alpha_g)$ and $(\alpha_1', \dots, \alpha_g')$ be two sets of attaching curves for U. Then the two sets of attaching curves can be connected by a series of isotopies and handleslides.

Thm Given two Heegaard diagrams $H_1 = (\Sigma_g, \{\alpha_1, \dots, \alpha_g\}, \{\beta_1, \dots, \beta_g\}, z_1)$ and $H_2 = (\Sigma_g, \{\alpha_1', \dots, \alpha_g'\}, \{\beta_1', \dots, \beta_g'\})$, $\exists$ a sequence of stabilizations, pointed isotopies, and pointed handleslides connecting them.

Q Given two Heegaard diagrams H_i for Y_i, $i = 1, 2$, how do I get a Heegaard diagram for the connected sum $Y_1 \# Y_2$?



We can also consider doubly-ptd Heegaard diagrams.

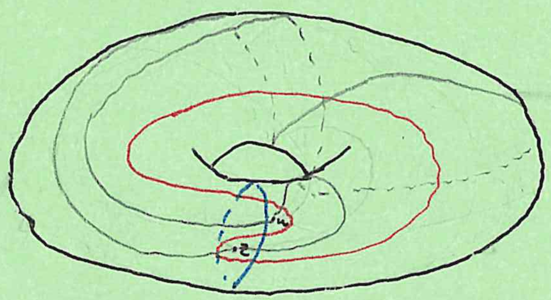


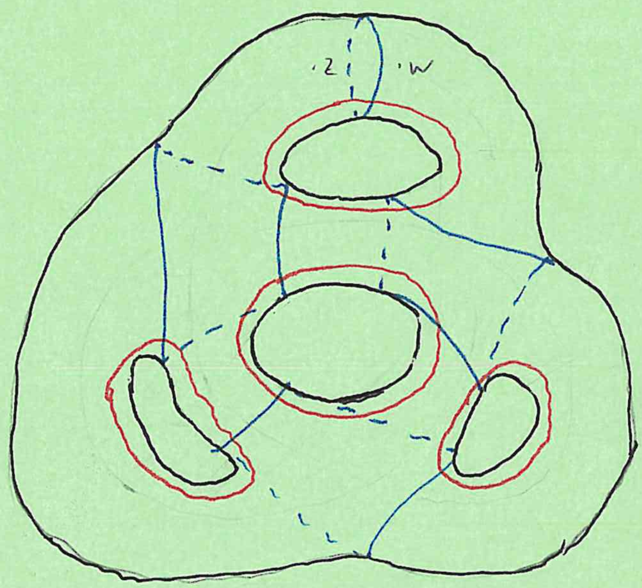
We write K as a union of a flowline from the index 0 critical pt to the index 3 critical pt. We put a basepoint z at the descending intersection and w at the ascending ".

For practical purposes, if we have a Heegaard diagram $H = (\Sigma, \vec{\alpha}, \vec{\beta}, w, z)$ we can recover the knot via.

- Connecting w to z by an arc in the complement of the $\vec{\beta}$ curves
- Connecting z to w by an arc in the complement of the $\vec{\alpha}$ curves.

<u>Exercise</u> This is necessarily unique up to isotopy.





We now ask for pointed isotopies and handleslides to avoid both basepoints.

<u>Q1</u> How do I reverse the orientation of the knot or link?

<u>Q2</u> How do I take the mirror of the knot or link?

<u>Q3</u> How do I get a Heegaard diagram for some surgery on the trefoil?

<u>Exercise</u> Which surgery is it?

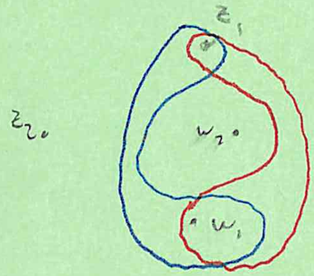
NB We can also take multi-pointed Heegaard diagrams.

$$H = (\Sigma_g, \{\alpha_1, \dots, \alpha_{g+k}\}, \{\beta_1, \dots, \beta_{g+k}\}, \{z_1, \dots, z_k\}) \text{ or}$$

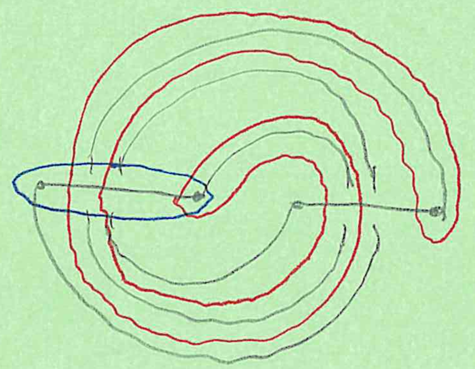
$$H = (\Sigma_g, \{\alpha_1, \dots, \alpha_{g+k}\}, \{\beta_1, \dots, \beta_{g+k}\}, \{z_1, \dots, z_k\}, \{w_1, \dots, w_k\})$$

eg)

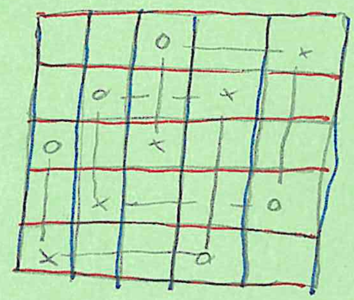
S^3)



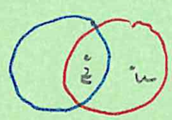
S^3)



$S^1 \times S^1$)



If we want to be able to change # of bps we allow the stabilization



Exercise Say I have $(\Sigma(K), \tilde{K})$ is a double branched cover. Construct

$$\begin{array}{c}(\Sigma(K), \tilde{K})\\ \downarrow \\ (S^3, K)\end{array}$$

an Heegaard diagram for $(\Sigma(K), \tilde{K})$.

Symmetric Products

Given a space Y, its g-fold symmetric product is

$$\mathrm{Sym}^g(Y) = \underbrace{Y \times \dots \times Y}_{g\text{-times}} \Big/ \underset{\text{symmetric group}}{S_g} = \{\text{unordered } g\text{-tuples of points in } Y\}$$

Example $\mathrm{Sym}^g(\mathbb{C}) \xleftrightarrow{\text{diffeomorphism}} \mathbb{C}^g$

$f(x) = (x-a_1)\dots(x-a_g) \longleftrightarrow$ coefficients of $f(x)$

Exercise $\mathrm{Sym}^g(S^1) \underset{\text{homeomorphic}}{\simeq} S^1$

Exercise $\mathrm{Sym}^g(\Sigma)$, for Σ a Riemann surface, is always a $2g$-dim'l complex mfd.

[Extremely not true in higher dimensions.]

Exercise The homotopy type of $\mathrm{Sym}^g(\Sigma - \{z\})$ is a skeleton of a torus. (In the standard CW decomposition.)

Inside $\mathrm{Sym}^g(\Sigma)$ we have:

Propn $\pi_2(\mathrm{Sym}^g(\Sigma)) = \mathbb{Z}$ for $g > 2$.

Propn $\pi_1(\mathrm{Sym}^g(\Sigma)) = H_1(\Sigma)$

- $\mathbb{T}_{\vec{\alpha}} = \alpha_1 \times \dots \times \alpha_g$
- $\mathbb{T}_{\vec{\beta}} = \beta_1 \times \dots \times \beta_g$

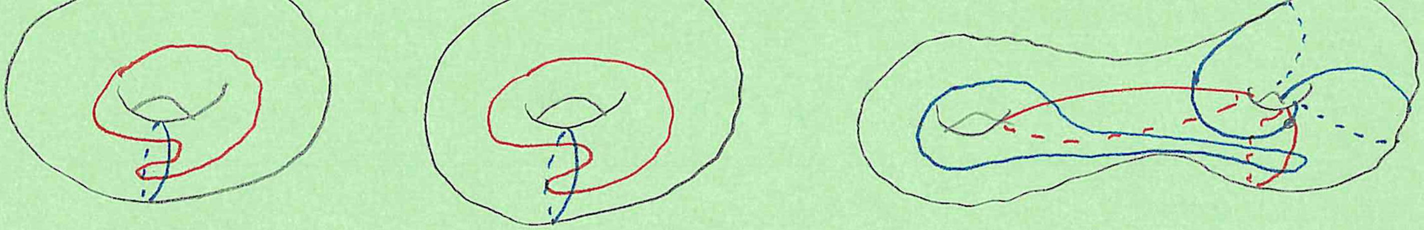
} This is an actual product since the α_i do not intersect

- $V_z = [z \times \mathrm{Sym}^{g-1}(\Sigma_g)]$
- $V_w = [w \times \mathrm{Sym}^{g-1}(\Sigma_g)]$

Exercise

$$\frac{H_1(\mathrm{Sym}^g(\Sigma))}{H_1(\mathbb{T}_\alpha) \oplus H_1(\mathbb{T}_\beta)} \simeq \frac{H_1(\Sigma)}{[\alpha_1], \dots, [\alpha_g], [\beta_1], \dots, [\beta_g]} \simeq H_1(Y)$$

We look at intersection points between $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ (Finitely many b/c $\mathbb{T}_{\vec{\alpha}} \pitchfork \mathbb{T}_{\vec{\beta}}$).



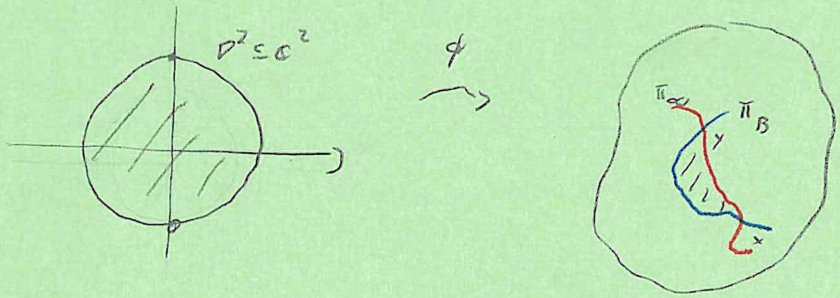
	R_1	R_2
B_1	1	1
B_2	1	2

<u>Def$_n$</u> $CF^-(Y) = \bigoplus_{x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \mathbb{F}_2[U]\langle x \rangle$

$\widehat{CF}(Y) = \bigoplus_{x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \mathbb{F}_2\langle x \rangle$

<u>Boundary Operator</u>

<u>Whitney Disk</u> A map $\phi : \{i, -i\} \to \mathbb{T}_\alpha \cap \mathbb{T}_\beta$



$Sym^g(\Sigma)$

s.t. $[-i, i] \to \mathbb{T}_{\vec{\alpha}}$. This is a Whitney disk from x to y.

$[i, -i] \to \mathbb{T}_{\vec{\beta}}$ The set of homotopy classes of such disks is $\pi_2(x, y)$.

- $\mathbb{T}_{\vec{\alpha}}, \mathbb{T}_{\vec{\beta}}$ are <u>totally real</u> in $Sym^g(\Sigma)$, i.e. $J(T(\mathbb{T}_{\vec{\alpha}})) \cap T(\mathbb{T}_{\vec{\alpha}}) = \phi$.

$$n_z(\phi) = \#_{Alg} \, Im(\phi) \cap V_z$$

$$n_w(\phi) = \#_{Alg} \, Im(\phi) \cap V_w.$$

Disks come w/ a natural multiplicative structure

$$\pi(x,y) * \pi(y,z) \longrightarrow \pi(x,z)$$

$$\pi_2'(Sym^g(\Sigma)) * \pi_2(x,y) \to \pi_2(x,y)$$

We can also study a disk via studying its <u>shadow</u>

<u>Defn</u> Let $D_1, \ldots, D_m$ denote the closures of the components of $\Sigma - \{\alpha_1, \ldots, \alpha_g\} - \{\beta_1, \ldots, \beta_g\}$. Given $\phi \in \pi_2(x,y)$ the shadow of ϕ, or domain associated to ϕ, is a formal linear combination of the regions $\{D_i\}_{i=1}^m$

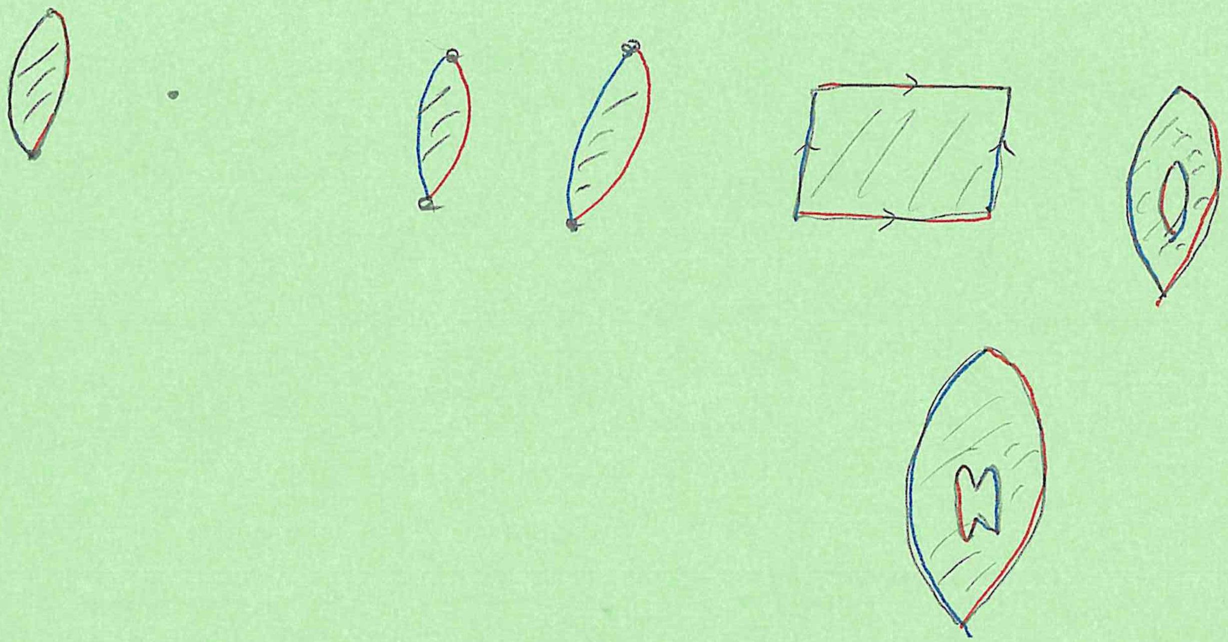
$$D(\phi) = \sum_{i=1}^m n_{z_i}(\phi) D_i)$$

For z_i some point in the interior of D_i.

The shadow is a (multiply branched) cover of the disk.

$$\begin{tikzcd} S \arrow[r] \arrow[d] & \Sigma \arrow[d, hook] \\ D^2 \arrow[r] & Sym^g(\Sigma) \end{tikzcd}$$

Things that are disks on a diagram of $g=2$



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Is there a disk connecting any two points of $\mathbb{T}_{\vec{\alpha}} \cap \mathbb{T}_{\vec{\beta}}$?