A (singly-pointed) Heegaard diagram consists of
\[ H = (\Sigma, \underline{a}, \underline{b}, z) \]

- \( \Sigma \) a surface of genus \( g \)
- \( \underline{a} = (a_1, \ldots, a_g) \) (respectively \( \underline{b} = (b_1, \ldots, b_g) \)) are a collection of nonintersecting simple closed curves on \( \Sigma \)
- \( g = \text{dim Span } \mathbb{Z} [a_i, b_j] \subseteq H_1(\Sigma; \mathbb{Z}) \) and \( \in H^1(\Sigma) \)
- \( z \) is a basepoint in \( \Sigma - \Sigma(a) - \Sigma(b) \).

A Heegaard diagram gives us a unique way of constructing a 3-manifold.

**Exercise**

- \( H_1(Y) = \mathbb{Z}_2 \)
- \( H_1(Y) = \mathbb{Z}/2 \mathbb{Z} \)
- \( H_2(Y) = \mathbb{Z} \)
- \( H_3(Y) = 0 \)

\[ \mathbb{Z}(2,3,5) \big/ \mathbb{Z}(2,3,7) \]
When do two Heegaard diagrams give the same 3-manifold?

When they are connected by Heegaard moves.

1. Isotopy of curves not crossing the basepoint.

2. Handleslides amongst $\alpha$ (respectively $\beta$) curves

$\alpha_1, \alpha_2, \alpha_3$ bound an embedded pair of pants in $\Sigma - \xi \alpha_1, \ldots, \alpha_3$

3. (Re)Stabilization

Connect sum with a genus 1 surface as shown.

The (Singer) given two Heegaard decompositions $(Y, \mathcal{U}, \mathcal{V}, \mathcal{W})$ and $(Y', \mathcal{U}', \mathcal{V}', \mathcal{W}')$ of a 3-manifold $Y$ having genus $g$ and $g'$ respectively, $Y$ at the $(k-g)$-fold stabilization of the first decomposition is diffeomorphic to the $(k-g')$-fold stabilization of the second decomposition.

Note that this has the effect of connect summing your manifold $\Sigma$.
Theorem. Let $N$ be a handlebody of genus $g$, and let $(\alpha_1, \ldots, \alpha_g)$ and $(\beta_1, \ldots, \beta_g)$ be two sets of attaching curves for $N$. Then the two sets of attaching curves can be connected by a series of isotopies and handle-slides.

Given two Heegaard diagrams $H_1 = (\Sigma_g, \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g)$ and $H_2 = (\Sigma_g, \alpha_1', \ldots, \alpha_g', \beta_1', \ldots, \beta_g')$, if a sequence of stabilizations, pointed isotopies, and pointed handle-slides connecting them.

Given two Heegaard diagrams $H_i$ for $y_i$, $i = 1, 2$, how do I get a Heegaard diagram for the connected sum $y_1 \# y_2$?

We can also consider doubly-paird Heegaard diagrams.

We write $K$ as a union of a flowline from the index 0 critical pt to the index 3 critical pt. We put a basepoint $z$ at the descending intersection and $u$ at the ascending $v$. 
For practical purposes, if we have a Heegaard diagram $H = \left( \mathcal{C}, \alpha, \beta, w, z \right)$ we can recover the knot via:

- Connecting $w$ to $z$ by an arc in the complement of the $\beta$ curves.
- Connecting $z$ to $w$ by an arc in the complement of the $\alpha$ curves.

Exercise: This is necessarily unique up to isotopy.

We now ask for pointed isotopies and handlebodies to avoid both basepoints.

Q1: How do I reverse the orientation of the knot or link?

Q2: How do I take the mirror of the knot or link?

Q3: How do I get a Heegaard diagram for some surgery on the trefoil?

Exercise: Which surgery is it?
We can also take multi-pointed Heegaard diagrams.

\[ H = \left( \Sigma, \xi, \ldots, \xi_{\gamma_k}, \xi_{\delta_1}, \ldots, \xi_{\delta_{\gamma_{k+1}}}, \xi_{\zeta_1}, \ldots, \xi_{\zeta_{\gamma_{k+1}}} \right) \]

\[ H = \left( \Sigma, \xi, \ldots, \xi_{\gamma_k}, \xi_{\delta_1}, \ldots, \xi_{\delta_{\gamma_{k+1}}}, \xi_{\zeta_1}, \ldots, \xi_{\zeta_{\gamma_{k+1}}} \right) \]

**Example**

\[ S^3 \]

**Exercise** Say I have \((\Sigma(K), \overleftarrow{v})\) is a double branched cover. Construct

\[ (S^3, K) \]

an Heegaard diagram for \((\Sigma(K), \overleftarrow{v})\).
Symmetric Products

Given a space $Y$, the symmetric product is

$$\text{Sym}^g(Y) = \frac{Y \times \ldots \times Y}{S_g}$$

where $S_g$ is the symmetric group.

**Example**

$$\text{Sym}^3(\mathbb{C}) \xrightarrow{f(x)} \mathbb{C}^3$$

with $f(x) = (x-a_1) \ldots (x-a_3)$.

**Exercise** $\text{Sym}^3(S^1) \approx S^1$

**Exercise** $\text{Sym}^3(\mathbb{R})$, for $\mathbb{R}$ a Riemann surface, is always a 2g-dimensional complex.

[Extremely not true in higher dimensions.]

**Exercise** The homotopy type of $\text{Sym}^3(\mathbb{R}^2)$ is a skeleton of a torus. (In the standard CW decomposition.)

Inside $\text{Sym}^g(\mathbb{R})$ we have:

- $\Pi^a_\mathbb{C} = \alpha_1 \times \ldots \times \alpha_g$
- $\Pi^a_\mathbb{R} = \beta_1 \times \ldots \times \beta_g$

- $V_\alpha = \left[ \mathbb{C} \times \text{Sym}^{g-1}(\mathbb{C}) \right]$  
- $V_\mu = \left[ \mathbb{R} \times \text{Sym}^{g-1}(\mathbb{R}) \right]$  

**Prop.** $H_2\left(\text{Sym}^3(\mathbb{C})\right) = \mathbb{Z}$ for $g \geq 2$.

**Prop.** $H_1(\text{Sym}^3(\mathbb{C})) = H_1(\mathbb{C})$.

Prove $H_1(\text{Sym}^3(\mathbb{C})) = H_1(\mathbb{C})$.

This is an actual product since the $\alpha_i$ do not intersect.

Exercise

$$H_1(\mathbb{C}) \otimes H_1(\mathbb{R}) \cong \mathbb{C} \times \mathbb{R}$$
We look at intersection points between $\pi_{A}$ and $\pi_{B}$

(Finitely many $b \in \pi_{A} \cap \pi_{B}$)

\[
\text{PeF}_{\pi_{A}} \leftarrow \text{CF}^{-}(Y) = \bigoplus_{x \in \pi_{A} \cap \pi_{B}} \mathbb{F}_{2} \langle u \rangle \langle x \rangle
\]

\[
\text{CF}(Y) = \bigoplus_{x \in \pi_{A} \cap \pi_{B}} \mathbb{F}_{2} \langle x \rangle
\]

Boundary Operator

\[\text{Whitney Disk} \quad \text{A map } q: \delta_{i} \cdot i^{3} \to \pi_{A} \cap \pi_{B}\]

$\Sigma$ $\mapsto$ $\pi_{A}$ $\cap$ $\pi_{B}$

This is a Whitney disk from $x$ to $y$.

$[i, \cdot i] \to \pi_{A}$

The set of homotopy classes of such disks is $\pi_{2}(\Sigma, y)$.

$\pi_{A}$, $\pi_{B}$ are totally real in $\text{Sym}^{3}(\Sigma)$, i.e. $\mathcal{F}(T(\pi_{A})) \cap T(\pi_{B}) = \emptyset$. 
\[ \eta^2 (q) = \varphi \text{ Im} (q) \cap V_2 \]
\[ \eta V (q) = \varphi \text{ Im} (q) \cap N V_w \]

Disks come with a natural multiplicative structure:

\[ \pi(x, y) \ast \pi(y, z) \rightarrow \pi(x, z) \]
\[ \pi_2' (\text{sym}^2(Z)) \ast \pi_2(x, y) \rightarrow \pi_2(x, y) \]

We can also study a disk via studying its shadow:

**Def.** Let \( \overline{D}_D \rightarrow \overline{D}_M \) denote the closures of the components of
\[ \Sigma = \Sigma^1 \cup \cdots \cup \Sigma^m \rightarrow \Sigma^D \cup \cdots \cup \Sigma^M \]. Given \( \phi \in \pi_2(x, y) \), the shadow of \( \phi \) or domain associated to \( \phi \) is a formal linear combination of the regions \( \overline{D}_i, i = 1, \ldots, m \):

\[ \mathcal{D}(\phi) = \sum_{i=1}^{m} \eta_2^i (\phi) \overline{D}_i \]

For \( z \) some point in the interior of \( D_i \).

The shadow is a (multiply branched) cover of the disk:

\[
\begin{array}{ccc}
S & \rightarrow & \Sigma \\
\downarrow & & \downarrow \\
D^2 & \rightarrow & \text{sym}^2(Z)
\end{array}
\]
Things that are disks on a diagram of $g=2$.

Is there a disk connecting any two points of $\Pi_0 \cup \Pi_1$?