A_∞ - algebras

Recall $\quad \mathbb{F} = \mathbb{Z}_2$

An A_∞ - algebra is an $\mathbb{F}$ - vectors space w/ maps $\mu_i : A^{\otimes i} \to A$ st $\forall \, x_1, \ldots, x_n \in A_i$, $\sum_{i,j} \mu_{n-j} (x_1, \ldots, x_{i-1}, \mu_{j+1}(x_i, \ldots, x_{i+j}), x_{i+j+1}, \ldots, x_n)) = 0.$

eg

① $\mu_1(\mu_1(x)) = 0 \Rightarrow \mu_1$ is a differential. $\quad \partial^2 = 0$

② $\mu_2(\mu_1(x), y) + \mu_2(x, \mu_1(y)) + \mu_1(\mu_2(x,y)) = 0 \quad$ Liebniz Rule

$$ \curlyvee + \curlyvee + \curlyvee = 0 $$

③ $\mu_1(\mu_3(x,y,z)) + \mu_3(\mu_1(x), y, z) + \mu_3(x, y, \mu_1(z)) + \mu_2(\mu_2(x,y), z) + \mu_2(x, \mu_2(y,z)) = 0$

$\Rightarrow \mu_3$ is a chain homotopy between $\mu_2(\mu_2(x,y), z)$ and $\mu_2(x, \mu_2(y,z))$.

(Multiplication associates up to homotopy).

$$ \curlywedge + \curlywedge + \curlywedge + \curlywedge + \curlywedge + \curlywedge = 0 $$

A_∞ - module

M: vector space over $\mathbb{F}$ w/ maps $m_{i+1} : M \times A^{\otimes i} \to M$ st

$$ 0 = \sum m_{n-i+1}(m_{i+1}(x, a_1, \ldots, a_i), a_{i+1}, \ldots, a_n) + \sum m_{n-j}(x, a_1, \ldots, a_{i-1}, \mu_j(a_{i+1}, \ldots, a_{i+j}), a_{i+j+1}, \ldots, a_n) $$

$$ \Big| \!\bullet\! \Big| = 0 \; , \quad \Big|\!\curlyvee + \Big|\!\curlyvee + \Big\backslash = 0 $$

Type D Structures

N: Vector spaces over $\mathbb{F}$ w/ $\delta: N \longrightarrow A \underset{\mathbb{F}}{\otimes} N$

ST $(\mu_2 \otimes \mathbb{I}_N) \circ (\mathbb{I}_A \otimes \delta_1) \circ \delta_1 + (\mu_1 \otimes \mathbb{I}_N) \circ \delta_1 = 0.$

$\leadsto \delta_k : N \longrightarrow A^{\otimes k} \otimes N$

<u>Defn</u> N is <u>bounded</u> if $\delta_k = 0$ for $k \gg 0.$

An <u>idempotent</u> is I st $m_i(x, \text{---}, I, \text{---}) = 0$ unless its $m(x, I) = x$

<u>Now</u> M an A_∞-module   Assume at least one of these is bounded
  N a type D structure   and $\mu_2(I, I) = I.$
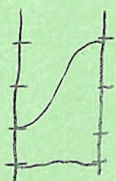
Let $M \boxtimes N = M \otimes N$ as a vector space.

$$\partial^\boxtimes (x \otimes y) = \sum_{k=0}^{\infty} (m_{k+1} \otimes \mathbb{I}_N) \circ (x \otimes \delta_k(y))$$

i.e.
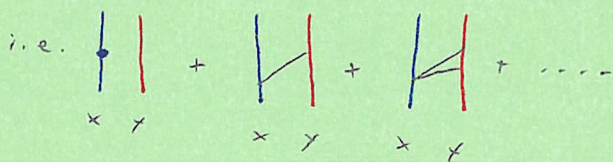
The specific A_∞-algebra we're interested in is the following.

Let $A(n,k) = \{ F: S \longrightarrow T : S, T \subseteq \{1, \dots, n\}, \; |S| = |T| = k, \; F(i) \ge i \; \forall i \}$
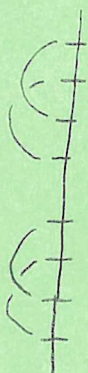
$= \{ k \text{ non-decreasing strands connecting } n \text{ vertically-arrang-} $
$\text{points} \}$

Example

$\in A(4,2)$

What's multiplication? Concatenation if you can and otherwise 0. Double crossings are set to 0.

We have a differential $d: A(n,k) \longrightarrow A(n,k) = \sum (\text{smoothing one-crossings})$
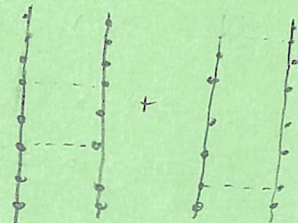
Now let Z be a pointed matched circle.

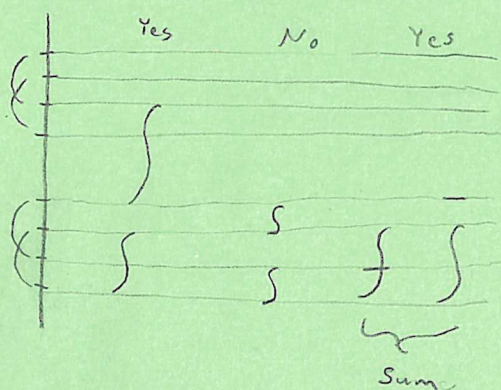$A(Z,i) \subseteq A(4n, i), \ 0 \leq i \leq 2n$. The example we care about most often is $A(4_n, n)$

• For each subset $\{s\}$ of $\{1, 2, \ldots, 2n\}$ we have an idempotent consisting of the sum of $I(s)$ For S any section of $\{1, \ldots, 4n\} \longrightarrow \{1, \ldots, 4n\}$ eg
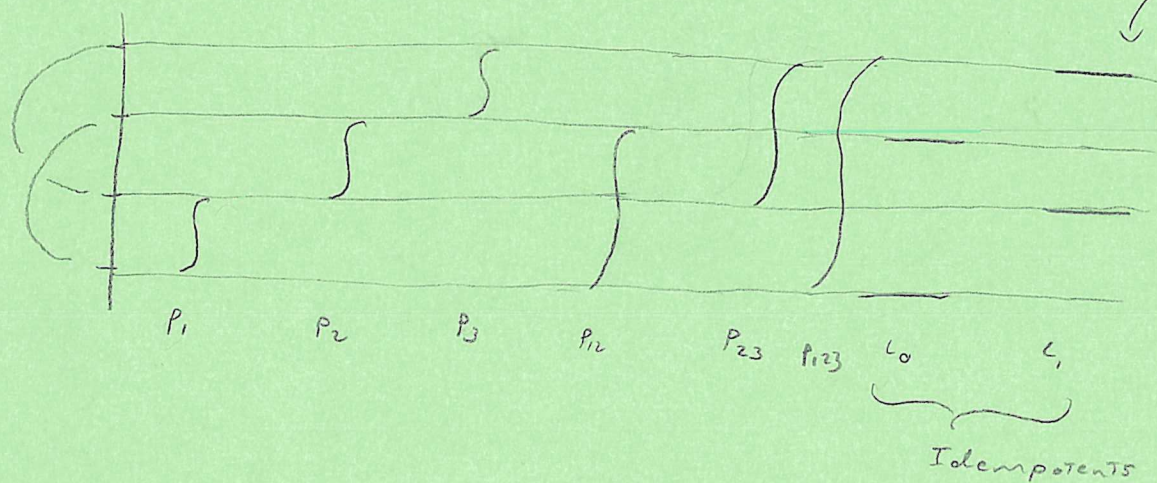
• For each set of upward-veering strands, w/ no two initial and no two final points paired, we consider the sum all ways of adding horizontal strands so that no two initial and final points are paired.

Examples

Yes   No   Yes

Sum

Special Case: Torus Boundary

Elements $A(Z, 1)$

This element is a sum.

$P_1 \quad P_2 \quad P_3 \quad P_{12} \quad P_{23} \quad P_{123} \quad \iota_0 \quad \iota_1$

$\underbrace{\qquad\qquad}_{\text{Idempotents}}$

$A(T^2)$

$P_1 P_2 = P_{12}$ etc

$\iota_0 P_1 = P_1$ etc

This is an honest algebra w/ $\mu_k = 0$ for $k \geq 3$.

Bordered Floer

• Let Σ a genus g surface w/ one puncture.

• $(\Sigma, \alpha_1^c, \ldots, \alpha_{g-1}^c, \beta_1, \ldots, \beta_g, \alpha_0^a, \alpha_1^a, z)$

• $\widehat{CFA}(\Sigma)$: Orient the boundary as $\partial \Sigma$
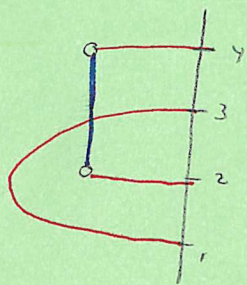
Generators

• g-tuples of points

     • One on every β circle

     • One on every α^c-circle

     • At most one on every α^a-circle

To each generator we have an associated idempotent (the one corresponding to the α-arc containing the generator).
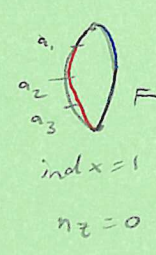
$$\widehat{CFA} = \mathbb{F} \langle generators \rangle$$

<u>Maps</u>
$$m_1(x) = \sum_{\alpha} \sum_{\substack{\phi \in \pi^2(x,y) \\ \mu(\phi) = 1 \\ n_z(\phi) = 0}} \# \widehat{m}(\phi)_y$$

$$m_{K+1}(x, a_1, \cdots, a_K) = \sum_{y} \sum_{x} \# \widehat{m}(\phi)_y \qquad a_1, \cdots, a_K \in t(F)$$

ind $x = 1$

$n_z = 0$

More generally one has to consider higher-genus curves w/ boundary punctures.

<u>Example</u>
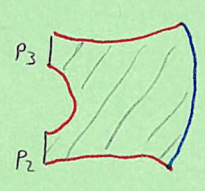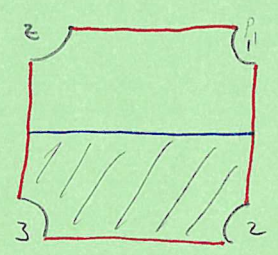
$$m_3(x, P_3, P_2) = x$$

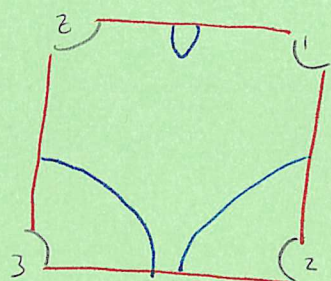$$m_{K+3}(x, P_3, P_{23}, \cdots, P_{23}, P_2) = x$$

↑ Not bounded...

We think of this as $S \longrightarrow \Sigma \times [0,1] \times \mathbb{R}$. ∂S has red and blue regions
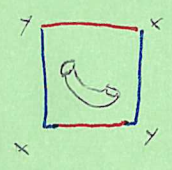
$x \longrightarrow +\infty$

$y \longrightarrow -\infty$

red $\longrightarrow \{0\}$

blue $\longrightarrow \{1\}$

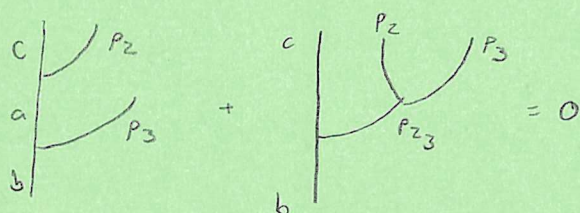holes $\longrightarrow$ Reeb chords

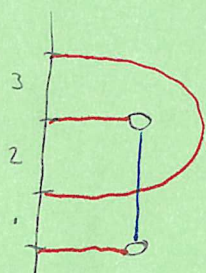Example

$$m_1(c) = b$$
$$m_2(c, P_2) = a$$
$$m_2(a, P_3) = b$$
$$m_2(c, P_{23}) = b$$

A∞ - relations satisfied

$$= 0$$

$\widehat{CFD}$ Z our oriented pointed matched circle $= -\partial \Sigma$

$\widehat{CFD}$ = a type D structure generated by F over
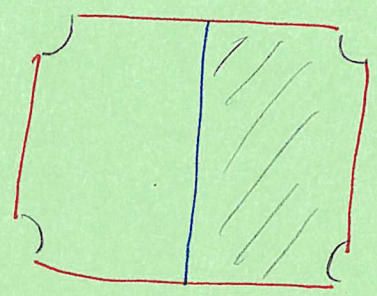   intersection points

generators ⤳ idempotent corresponding to the arc
which is not occupied.

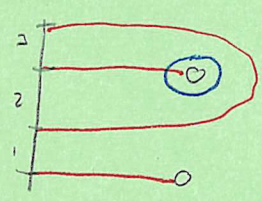$$\delta_1(x) = \sum_7 \sum_x \# \widehat{m}(\phi)(a_1, a_2, \ldots a_n) \otimes \gamma$$
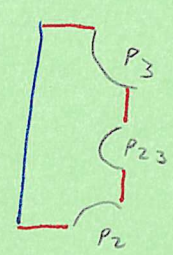

ind ±1,
$n_z = 0$

Example

$$\partial_1(x) = P_2 P_3 \otimes x$$
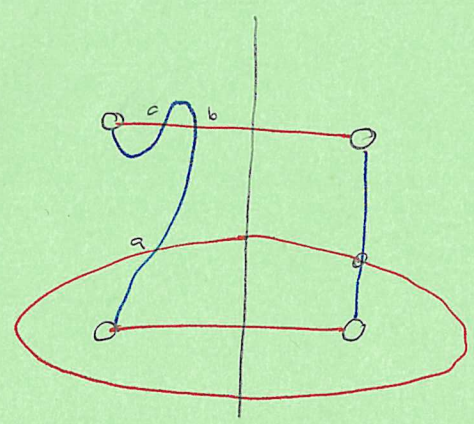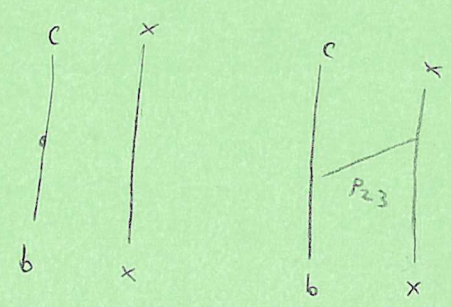$$= P_{23} \otimes x$$

What about

$P_2 P_{23} P_3 = 0$ so this disk doesn't count. Hence $\widehat{CFD}$ computes fewer disks than $\widehat{CFA}$.

Example

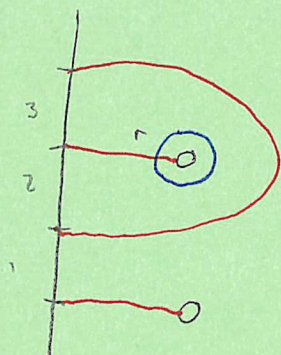Consider $\widehat{CFA} \boxtimes \widehat{CFD}$

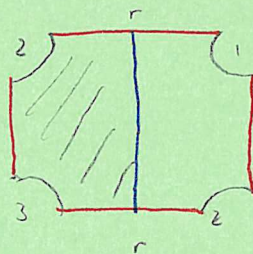$$\partial(c \boxtimes x) = 2(b \boxtimes x) = 0$$

Example The surgery triangle in bordered.

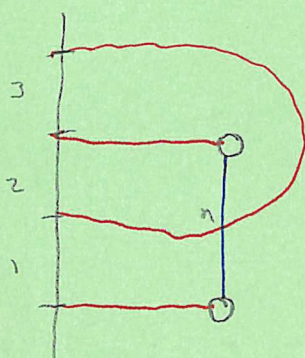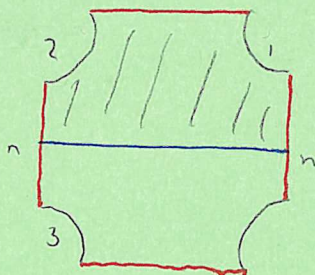Split Y as Y_1 w/ $\partial Y_1 = S^1 \times S^1$, and a solid torus.

$\boxed{\mathcal{H}_{-1}}$

$$\partial(r) = P_{23} r$$

$\boxed{\mathcal{H}_0}$

$$\partial(n) = P_{12} n$$

$\boxed{\mathcal{H}_\infty}$
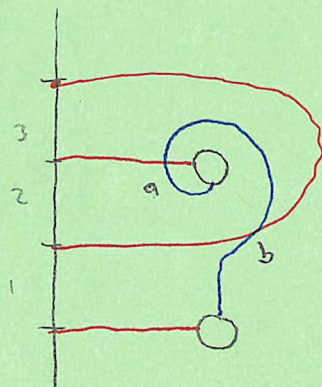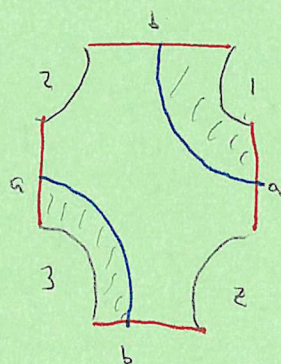

$$\partial a = P_3 b + P_1 b$$

There is a short exact sequence:

$$0 \longrightarrow \widehat{CFD}(\mathcal{H}_\infty) \overset{\phi}{\longrightarrow} \widehat{CFD}(\mathcal{H}_{-1}) \overset{\psi}{\longrightarrow} \widehat{CFD}(\mathcal{H}_0) \longrightarrow 0$$

$$r \longmapsto b + P_2 a$$

$$a \longmapsto n$$
$$b \longmapsto P_2 n$$

Given any Y_1 w/ torus boundary, tensoring $\widehat{CFA}(Y_1)$ with this ses gives the surgery exact triangle.

[Note: These are morphisms of type D modules

Note $\partial(\phi(r)) = \partial(b + P_2 a) = P_{23} b$

$\phi(\partial r) = \phi(P_{23} r) = P_{23}(b + P_2 a) = P_{23} b$

Also $\partial(\psi(a)) = \partial n = P_{12} n$

$\psi(\partial a) = \psi(P_3 b + P_1 b) = P_{12} n$

$\partial(\psi(b)) = \partial P_2 n = 0,$

$\psi(\partial b) = \psi(0) = 0,$

[compatibility: $(\mu_2 \otimes \mathbb{I}_N) \circ (\mathbb{I}_A \otimes \delta^1) \circ \delta^1 + (\mu_1 \otimes \mathbb{I}_N) \circ \delta^1 : N_1 \longrightarrow A \otimes N_2$ vanishes.]

A homomorphism of type-D structures N_1, N_2 comes from a map

$\psi_1 : N_1 \longrightarrow A \otimes N_2$, which can be used to construct maps $\psi_K : N_1 \longrightarrow A^{\otimes k} \otimes N_2$

The condition is that $(\mu \otimes \mathbb{I}_{N_2}) \circ \psi = 0.$]

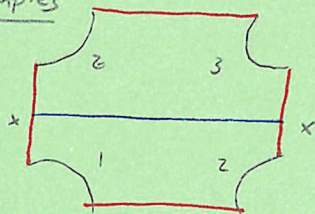$$\psi_K = \sum_{i+j=K-1} (\mathbb{I} \circ \delta_{N_1}^i) \otimes (\quad\quad)$$

The pairing theorem

Let $Y \cong Y_1 \underset{F}{\cup} Y_2$. Then $\widehat{HF}(Y_1 \underset{F}{\cup} Y_2) \cong \widehat{CFA}(Y_1) \boxtimes \widehat{CFD}(Y_2)$
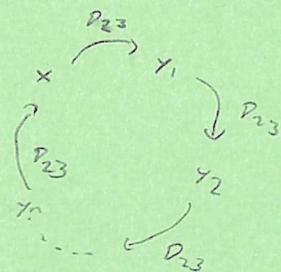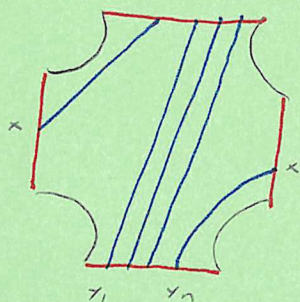
Let's think specifically about $\widehat{CFD}(Y)$ for a while.

Examples

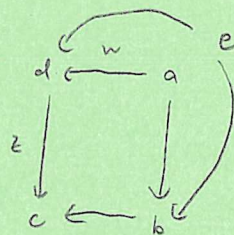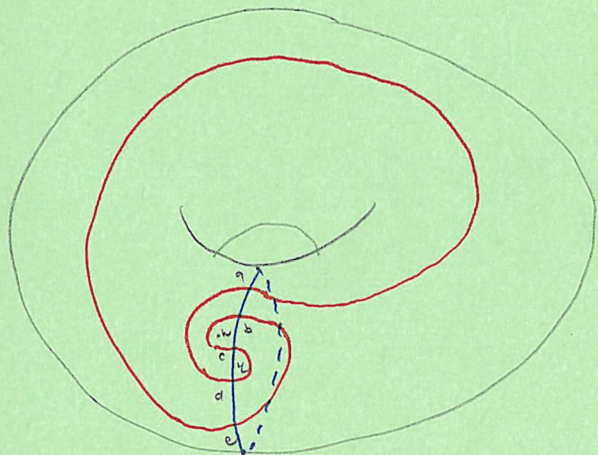
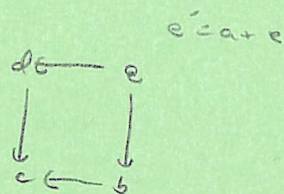$\partial x = P_{12} x$

$x \circlearrowright P_{12}$

Let $Y = S^3 - \text{nbd}(K)$ w/ framing n. Algorithm $CFK^\infty(K) \rightsquigarrow \widehat{CFD}(Y)$.
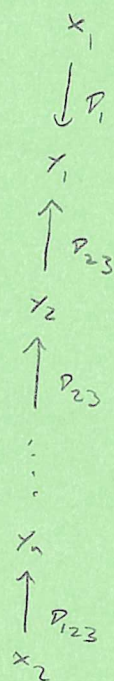
Example CFK^∞

$\otimes \; \mathbb{Z}[U, U^{-1}]$

We can replace by a vertically simplified basis:

$$d\epsilon \longleftarrow e$$
$$e' = a + e$$
$$\downarrow \qquad \downarrow$$
$$c \longleftarrow b$$

From CFK° to $\widehat{CFD}(Y)$: Take a vertically simplified basis. To a vertical arrow of length ℓ we associate:

$$i_0 x_j = i_1 y_j$$

$$\begin{array}{c} x_1 \\ \downarrow e \\ x_2 \end{array}$$

$$\begin{array}{c} x_1 \\ \downarrow D_1 \\ y_1 \\ \uparrow D_{23} \\ y_2 \\ \uparrow D_{23} \\ \vdots \\ y_n \\ \uparrow D_{123} \\ x_2 \end{array}$$

Given a horizontally simplified basis, to a horizontal arrow of length ℓ $x_2' \longleftarrow x_1'$ we associate the chain $x_2' \xleftarrow{D_2} y_\ell' \xleftarrow{D_{23}} \cdots \xleftarrow{D_{23}} y_1' \xleftarrow{D_3} x_1'$

Finally between x_v and x_h the distinguished elements in their respective bases we have the _unstable chain_ which depends on the framing.
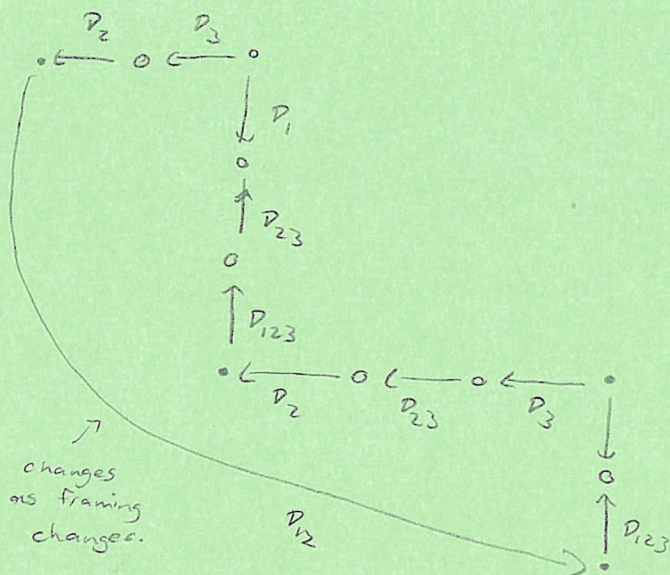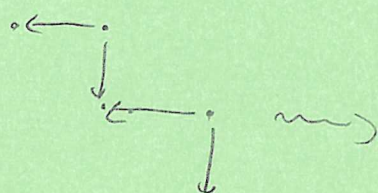
When $n < 2\tau(K)$, $x_v \xrightarrow{D_1} z_1 \xleftarrow{D_{23}} z_2 \cdots \xleftarrow{D_{23}} z_m \xleftarrow{D_3} x_h \qquad m = 2\tau(K) - n$

$n = 2\tau(K) \qquad x_v \longrightarrow x_h$

$n > 2\tau(K) \qquad x_v \xrightarrow{D_{123}} z_1 \xrightarrow{D_{23}} \cdots \longrightarrow z_m \xrightarrow{D_2} x_h \qquad m = |2\tau(K) - n|$
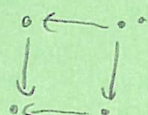
Example

$K = T_{3,4}$ $CFK^\infty(K)$

changes as framing changes.

D_2 D_3 D_1 D_{23} D_{123} D_2 D_{23} D_3 D_{12} D_{123}

Example

$CFK^\infty(K)$ $\widehat{CFD}(Y)$

D_2 D_3 D_1 D_1 D_{123} D_{123} D_2 D_3 D_{23}