Today/Monday: A survey of various concordance invariants and their properties.

Recall we have $K \mapsto \text{CFK}^{\infty}(K)$, a filtered, $\mathbb{Z}$-graded complex over $\mathbb{F}[u^\pm 1]$.

- $H_*\left(\mathbb{C}\mathbb{F} \mathbb{P}^3\right) \cong \text{HF}^+\left(S^3\right)$
- $H_*\left(\mathbb{C}\mathbb{F} \mathbb{P}^2\right) \cong \text{HF}^0\left(S^3\right)$
- $H_*\left(\mathbb{C}\mathbb{F} \mathbb{P}^3\right) \cong \text{HF}^+\left(S^3\right)$
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We can look at the maps $D$ and $V$ as well.

**Concordance Invariants**

We've already discussed $\tau$:

$$\tau(K) = \min \{ s \in \mathbb{Z} : \mathbb{C}\mathbb{F} \mathbb{P}^3 \to C^{\mathbb{F} \mathbb{P}^3} \text{ induces a surjection on homology} \}$$

$$\tau(K) = -\max \{ s \in \mathbb{Z} : \mathbb{F} \mathbb{P}^3 \to \text{HF}^{-}(K) \}$$

For all $n \in \mathbb{N}$, $\text{HF}^{0}(\mathbb{Z})$.

From the first definition, $\tau(K \# K) = \tau(K) + \tau(K)$

- $\tau(K) = -\tau(K)$
- $\tau(K) = 0$ if $K$ is slice.
Thm. \( \eta : \mathbb{Z} \to \mathbb{Z} \) is a surjective homomorphism.

- \( \eta'(K) \leq \eta(K) \)
  - From Pongsoo's talk

- If \( K \) is (quasi-)alternating, \( \eta(K) = -\frac{\sigma(K)}{2} \)

- For the \( p,q \) torus knot, \( p, q \geq 1 \) coprime, \( \eta(T_{p,q}) = \frac{p-1)(q-1)}{2} \), \( \eta(T_{p,q}) \).

- Under crossing change \( K_+ \to K_- \), we have
  \[ \eta(K_-) = 1 \leq \eta(K_+) \leq \eta(K_+) \]

Note: \( \eta \) is analogous to \( s \) from Khovanov homology. \( \frac{s(K)}{2} \) also has the properties above. But they are not the same. [Hedden-Kriz]

Other concordance invariants

From surgery maps

Recall we let \( W_N(K) \) be the two-handle cobordism from \( S^3 \) to \( S^3_N(K) \) for \( N \gg 0 \), and consider the cobordism map associated to \( -W_N(K) \) from \( S^3_N(K) \) to \( S^3 \).

Recall further that the map

\[ y_N : \mathbb{C} \otimes \max(e,j-s) \otimes 0 \otimes 3 \to \mathbb{C} \otimes e \otimes 0^3 \]

is the cobordism map associated to the spin\(^c\) structure \( e \) on \( W_N \langle e, (t), [\Sigma] \rangle = 2s \) as long as \( 1s | N \). Here \( \Sigma \) is a capped-off Seifert surface, \( y_N : \text{HF}^- (S^3_N(K), s) \to \text{HF}^- (S^3) \cong \text{IF} [U] \)

Likewise there is a hat version

\[ \hat{y}_N : \mathbb{C} \otimes \max(e,j-s) = 0 \otimes 3 \to \mathbb{C} \otimes e \otimes 0^3 \]

\[ \hat{y}_N : \text{HF}^- (\hat{S}_N(K), s) \to \text{HF}^- (S^3) \cong \text{IF} \]
We let \( r(K) = \min S: v_{S, x} \) is surjective.\(^3\)

Note that \( v_{S, x} \) is trivial for \( S < r(K) \) and nontrivial for \( S \geq r(K) \), so \( r(K) \) is either \( r(K) \) or \( r(K) + 1 \).

**Examples**

\[
\begin{align*}
\varepsilon(K) &= 1 \\
S &= 1 \\
\varepsilon(K) &= 1
\end{align*}
\]

\[
\begin{align*}
\varepsilon(K) &= -1 \\
S &= -1 \\
\varepsilon(K) &= 0
\end{align*}
\]

\[
\begin{align*}
\varepsilon(K) &= 0
\end{align*}
\]

We also have \( v^{-}(K) \), which is the \( \min S: v_{S, x}^{*-} \) is surjective.\(^3\)

This has the relationship:

\[
\varepsilon(K) \leq r(K) \leq v^{-}(K) \leq g_{y}(K)
\]

This gap can be made as large as you like.\(^3\)

Recall

\[
V_{0} = \frac{1}{2} d_{x}^{2}
\]

Note that \( v^{-} = \min S: v_{S} = 0 \).\(^3\)

\[
V_{S} = \text{rank}(\coker v_{S, x}^{*-})
\]
\[ [\varepsilon_1] \text{ generates } \quad H^1(C_0) \cong 0 \]

\[ [\varepsilon_2] \text{ dies } \]

\[ \text{but } \]

\[ \varepsilon_{\leq 2} \text{ persists into } A_1. \]

So \( V_1 = 1, \ V_2 = 0. \)

\[ \xi_0 \rightarrow \xi_1 \rightarrow \xi_2 \rightarrow \xi_3 \quad (\ast) \]

So \( V^{-}(K) = 2 \)

\[ \hat{V}(K) = ? \text{ Vertical homology is generated by } \varepsilon_0; \text{ it has an outgoing arrow. So we see this is } 1. \]

Four ball genus bounds and a refinement of the Ozsv\'ath-Szab\'o can improve.
We also have the $e$-invariant, defined as follows:

**Simple Defn**

- IF $v(k) = e(k)+1$, then $e(k) = 1$.
- IF $v(k) = e(k)$ and $v(-k) = e(-k)$, then $e(k) = 0$.
- IF $v(-k) = e(-k)+1$, then $e(k) = 1$.

This is exhaustive but not illuminating.

More to the point: we say a basis for $CFK^\infty$ is **vertically simplified** if for each $x_i$ exactly one of the following is true:

- $\exists x_i = x_j$ for some $x_j$.
- $x_i$ is in the kernel but not the image of $\partial_v$.
- $x_i = \partial_v x_j$ for a unique $x_j$.

Similarly a **horizontally simplified** basis.

**Lemma** A horizontally simplified basis for $CFK^\infty$ is some $x_j$ is the distinguished generator of a vertically simplified basis.

- $e(k) = 1$ if $x_j$ is not in the kernel of the horizontal differential.
- $e(k) = 0$ if $x_j$ is in the kernel but not the image of the horizontal differential.
- $e(k) = 1$ if $x_j$ is in the image of the horizontal differential.
\[ \varepsilon(k) = -1. \]

**Properties of \( \varepsilon \)**

1. If \( k \) is slice, \( \varepsilon(k) = 0 \).
2. If \( \varepsilon(k) = 0 \), \( \varepsilon(\overline{k}) = 0 \).
3. \( \varepsilon(k) = -\varepsilon(-k) \).
4. For \( k \) homologically thin, \( \varepsilon(k) = \text{sgn}(\bar{\varepsilon}(k)) \).
5. If \( \bar{\varepsilon}(k) = 1 \) or \( 0 \), \( \varepsilon(k) = \text{sgn} \bar{\varepsilon}(k) \).
6. If \( \varepsilon(k) = \varepsilon(k',-k') \), then \( \varepsilon(k, k') = \varepsilon(k, k') \).

Was used to give the first proof of

**Thm (Hom) The group \( G \) contains a direct summand isomorphic to \( \mathbb{Z}^\infty \).**

**Idea of this proof**: One looks at the group

\[ EF_k := \left( \text{CFR}^{\infty}(k) : k \in \mathbb{Z}^2, \overline{\mathbb{Z}} \right) / \sim \]

where \( c \ll c' \) if \( \varepsilon(c, c') = 0 \). This group is ordered, \( [c, l] \triangleright [c, k] \triangleright [c,] \).

\( \varepsilon(c, c') = 1 \). The proof relies on finding classes \( [c,] \triangleright [c,] \triangleright [c,] \) (this means \( n \triangleright n \triangleright n \)).

Consider the dictionary order on the plane...