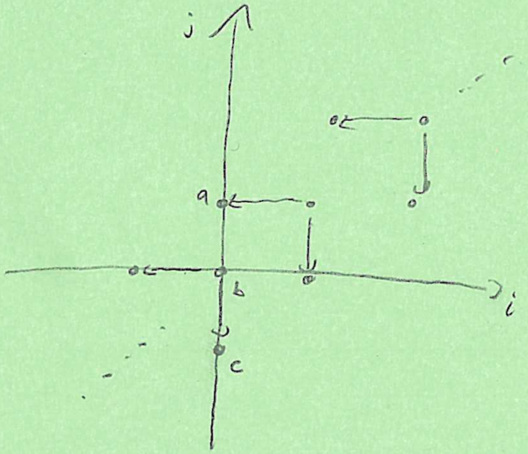
CFK^∞ and concordance invts

Today / Monday A survey of various concordance invts and their properties.

Recall we have $K \rightsquigarrow CFK^\infty(K)$ $\mathbb{Z} \oplus \mathbb{Z}$ filtered, $\mathbb{Z}$ graded complex over $\mathbb{F}[U, U^{-1}]$.



$$H_*(C\{i \geq 0\}) \simeq HF^+(S^3)$$

$$H_*(C\{i = 0\}) \simeq \widehat{HF}(S^3)$$

$$H_*(C\{i \leq 0\}) \simeq HF^-(S^3)$$

$$H_*(\mathcal{F}_j(C\{i=0\})) \simeq \widehat{HFK}(S^3, K)$$

$$H_*(\mathcal{F}_j(C\{i \leq 0\})) \simeq HFK^-(S^3, K)$$

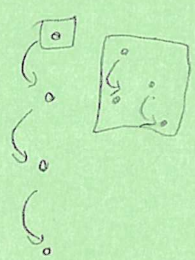
We can look at the maps ∂^{vert} and ∂^{horz} as well.

Concordance Invariants

We've already discussed τ:

$\tau(K) = \min\{s : i : C\{i=0, j \leq s\} \to C\{i=0\}$ induces a surjection on homology$\}$

$\tau(K) = -\max\{s \in \mathbb{Z} : \exists\ \theta \in HFK^-(K, s)$ st for all $n \in \mathbb{N}$, $U^n\theta \neq 0\}$



From the first defn
- $\tau(K_1 \# K_2) = \tau(K_1) + \tau(K_2)$
- $\tau(\overline{K}) = -\tau(K)$
- $\tau(K) = 0$. if K is slice.

(Ozsváth-Szabó)

Thm • $\tau : \mathcal{C} \to \mathbb{Z}$ is a surjective homomorphism.

• $|\tau(K)| \leq g_4(K)$

Dongsoo's talk

• If K is (quasi-)alternating, $\tau(K) = -\frac{\sigma(K)}{2}$

• For the p,q torus knot, w $p,q \geq 1$ coprime, $\tau(T_{p,q}) = \frac{(p-1)(q-1)}{2} = g(T_{p,q})$.

• Under crossing change $K_+ \to K_-$, we have

$$\tau(K_+) - 1 \leq \tau(K_-) \leq \tau(K_+)$$

Note τ is analogous to s from Khovanov homology. $\frac{s(K)}{2}$ also has the properties above. **But** they are not the same [Hedden-Ording]

Other concordance invts

From surgery maps

Recall we let $W_N(K)$ be the two handle cobordism from S^3 to $S^3_N(K)$ for $N >> 0$, and consider the cobordism map associated to $-W_N(K)$ from $S^3_N(K)$ to S^3.

Recall further that the map

$$v_s^- : C\{\max(i, j-s) \leq 0\} \to C\{i \leq 0\}$$

is the cobordism map associated to the spin^c-structure t on W st $\langle c_1(t), [\hat{F}] \rangle = 2s$, as long as $|s| < N$. Here F is a capped-off Seifert surface. $v^-_{s,*} : HF^-(S^3_N(K), s) \to HF^-(S^3) \simeq \mathbb{F}[U]$

Likewise there is a hat version

$$\hat{v}_s : C\{\max(i, j-s) = 0\} \to C\{i = 0\}$$

$$\hat{v}_{s,*} : \widehat{HF}(S^3_N(K), s) \to \widehat{HF}(S^3) \simeq \mathbb{F}$$

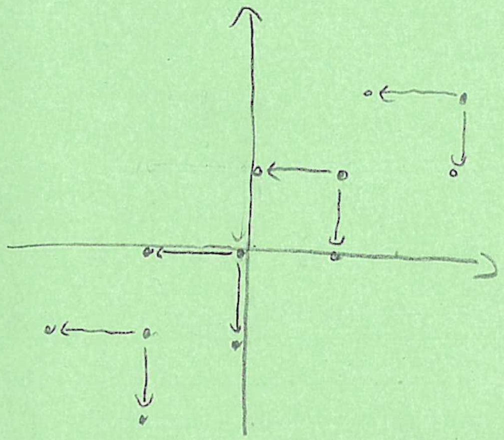
We let $r(K) = \min\{s : \hat{v}_{s,*} \text{ is surjective}\}$.

Note that $\hat{v}_{s,*}$ is trivial for $s<\tau(K)$ and nontrivial for $s>\tau(K)$;

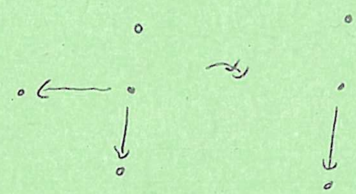
so $r(K)$ is either $\tau(K)$ or $\tau(K)+1$.

Examples

$\tau(K) = 1$

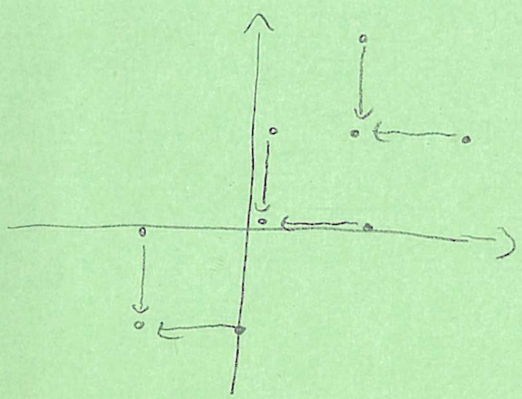


$s=1$

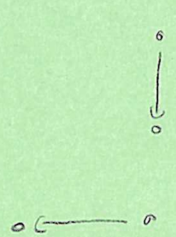


✓ $r(K) = 1$

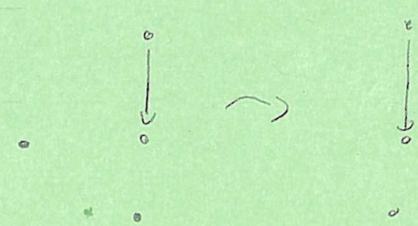
$\tau(K) = -1$



$s=-1$



$s=0$



$V(K) = 0.$

We also have $v^-(K)$, which is the $\min\{s : v^-_{s,*} \text{ is surjective}\}$

This has the relationship

$$\tau(K) \le r(K) \le v^-(K) \le g_4(K)$$

This gap can be made as large as you like.

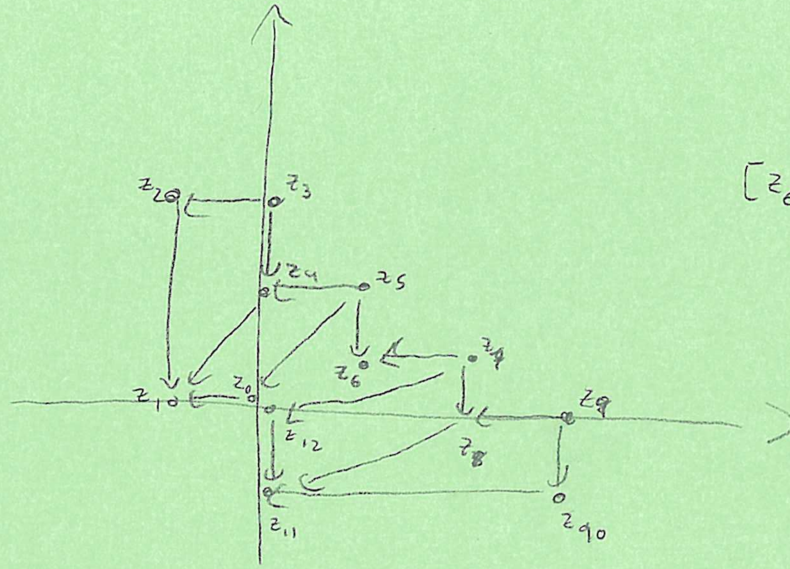
$K = T_{2,5} \# 2T_{2,3} \# -T_{2,3;2,5}$

$K_{p,3p-1}$

Recall
$V_0 = \frac{-1}{2} d(S^3_1($

$V_s = \operatorname{rank}(\operatorname{coker} v^-_{s,*})$

Note that $v^- = \min\{s : V_s = 0\}$.



$[z_6]$ generates

$H_*(C\{i \geq 0\})$

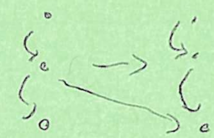
$[Uz_6]$ dies

<u>but</u>

$[Uz_6]$ persists into

A_1^+.

So $V_1 = 1$. $V_2 = 0$.



So $\nu^-(K) = 2$

$\hat{\nu}(K) = ?$ Vertical homology is generated by z_0; it has an outgoing arrow. So we see this is 1.

Four ball genus bounds and a refinement of the Ozsváth–Szabó tau invt

We also have the ε-invariant, defined as follows:

Simple Defn

- If $\nu(K) = \tau(K)+1$, then $\varepsilon(K) = -1$.
- If $\nu(K) = \tau(K)$ and $\nu(-K) = \tau(-K)$, then $\varepsilon(K) = 0$.
- If $\nu(-K) = \tau(-K)+1$, then $\varepsilon(K) = 1$.

This is exhaustive but not illuminating.

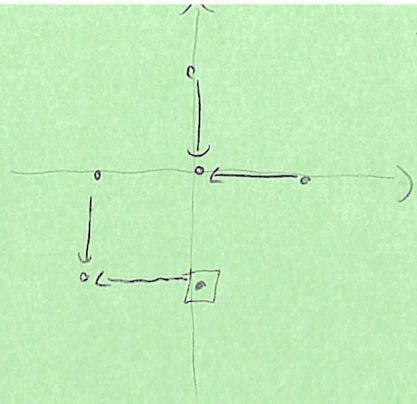
More to the point: we say a basis for CFK^∞ is vertically simplified if for each x_i exactly one of the following is true:

- $\partial^{vert} x_i = x_j$ for some x_j
- x_i is in the kernel but not the image of ∂^{vert}
- $x_i = \partial^{vert} x_j$ for a unique x_j

Similarly a horizontally simplified basis.

Lemma (Hom) $\exists$ a horizontally simplified basis for CFK^∞ s.t. some x_j is the distinguished generator of a vertically simplified basis.

- $\varepsilon(K) = -1$ if x_j is not in the kernel of the horizontal differential
- $\varepsilon(K) = 0$ if x_j is in the kernel but not the image of the horizontal differential
- $\varepsilon(K) = 1$ if x_j is in the image of the horizontal differential.



$\varepsilon(K) = -1.$

Properties of ε

1. If K is slice, $\varepsilon(K) = 0$.
2. If $\varepsilon(K) = 0$, $\tau(K) = 0$.
3. $\varepsilon(K) = -\varepsilon(-K)$
4. For K homologically thin, $\varepsilon(K) = \mathrm{sgn}(\tau(K))$.
5. If $g(K) = |\tau(K)|$, $\varepsilon(K) = \mathrm{sgn}\,\tau(K)$.
6. If $\varepsilon(K_1) = \varepsilon(K_2)$, then $\varepsilon(K_1 \# K_2) = \varepsilon(K_1)$. If $\varepsilon(K_1) = 0$, $\varepsilon(K_1 \# K_2) = \varepsilon(K_2)$

Was used to give the first proof of

Thm (Hom) The group $\mathcal{C}_{TS}$ contains a direct summand isomorphic to $\mathbb{Z}^\infty$.

Idea of this proof: One looks at the group

$$\mathcal{CFK} := (\{CFK^\infty(K) : K \subseteq S^3\}, \#)/\sim_\varepsilon$$

where $C_1 \sim_\varepsilon C_2$ if $\varepsilon(C_1 \otimes C_2^*) = 0$. This group is ordered, $[C_1] > [C_2] \Leftrightarrow$ $\varepsilon(C_1 \otimes C_2^*) = 1$. The proof relies on finding classes $[C_i]$ st $[C_{i+1}] \gg [C_i]$ (this means $\nexists\, n$ st $n[C_i] > [C_{i+1}]$.)

[Consider the dictionary order on the plane...]