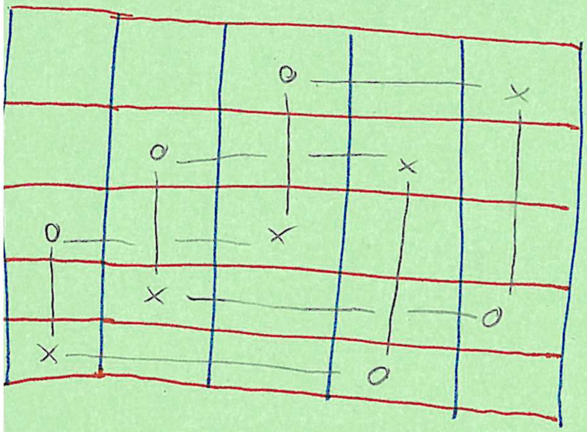
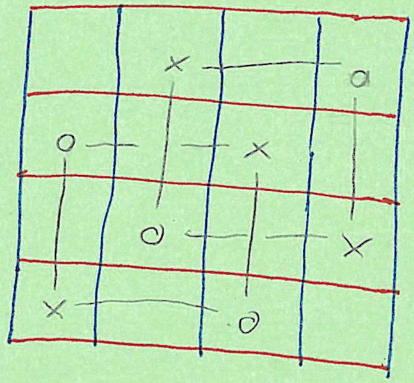
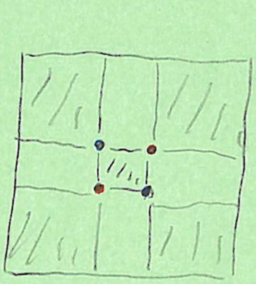
Lecture 18

Last Time Grid Diagrams





- How do you get a mirror? A reverse?
- Homology
  - Generators are the $n!$ states, corresponding to elements of S_n.
  - Differentials connect states that differ in two places.
    - $\mathrm{Rect}(\vec{x},\vec{y}) = \{$Set of rectangles from $\vec{x}$ to $\vec{y}\}$
    - $\mathrm{Rect}^\circ(\vec{x},\vec{y}) = \{$Set of empty rectangles from $\vec{x}$ to $\vec{y}\}$

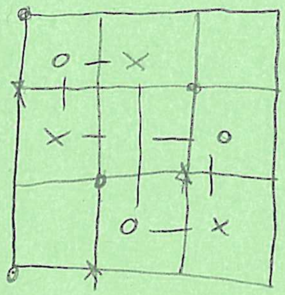


$$\partial \vec{x} = \sum_{y \in S(G)} \sum_{\substack{r \in \mathrm{Rect}^\circ(\vec{x},\vec{y}) \\ r \cap \mathbb{O} = 0 \\ r \cap \mathbb{X} = 0}} \vec{y}$$

- **Result** $\widetilde{HFK}(S^3, K) = \widehat{HFK}(S^3, K) \otimes V^{\otimes(n-1)}$

$V = \mathbb{F}_2 \oplus \mathbb{F}_2$

Example



$$\begin{array}{ccc} & (123) & \\ \swarrow & \downarrow & \searrow \\ (213) & (132) & (321) \end{array} \qquad \begin{array}{c} (312) \\ \\ (231) \end{array}$$

Gradings

$M_{\mathbb{O}}(\vec{x}^{NWO}) = 0$ } Exercise: why is this correct?

$$M_{\mathbb{O}}(\vec{x}) - M_{\mathbb{O}}(\vec{y}) = 1 - 2\#(r \cap \mathbb{O}) + 2\#(\vec{x} \cap \mathrm{Int}(r))$$

- $M(\vec{x}) = M_{\mathbb{O}}(\vec{x})$
- $A(\vec{x}) = \frac{1}{2}(M_{\mathbb{O}}(\vec{x}) - M_{\mathbb{X}}(\vec{x})) - \frac{n-1}{2}$

or $A(\vec{x}^{NWO}) = \tau$; $A(\vec{x}) - A(\vec{y}) = \#(r \cap \mathbb{X}) - \#(r \cap \mathbb{O})$.

This can be written as an explicit non-relative function:

For two subsets of the plane, let $\mathcal{I}(S_1, S_2) = \#\left\{ \begin{array}{l} (x_1,y_1) \in S_1 \\ (x_2,y_2) \in S_2 \end{array} : \begin{array}{l} x_1 < x_2 \\ y_1 < y_2 \end{array} \right\}$

$$M(S_1, S_2) = \mathcal{I}(S_1, S_1) - \mathcal{I}(S_1, S_2) - \mathcal{I}(S_2, S_1) + \mathcal{I}(S_2, S_2)$$

$$M(S) = M(S, \mathbb{O}) + 1$$

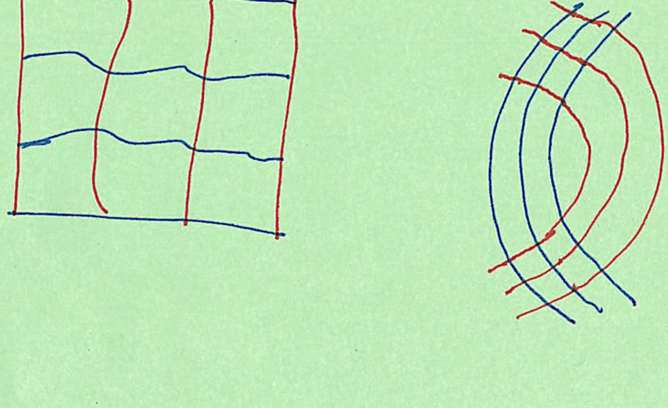
$$A(S) = \frac{1}{2} M(S, \mathbb{O}) - M(S, \mathbb{X}) - \frac{n-1}{2}$$

Can also write the Alexander grad... as a windin...

Boundary-injectivity Suppose we have a class $\phi \in \pi_2(\vec{x}, \vec{y})$ and a $J(t)$ hol'c representative $u: D^2 \to Sym^g(\Sigma)$. Consider the map $\hat{u}: F \to \Sigma$ induced by u (this goes from a branched cover of D^2 to Σ, and has image $D(\phi)$). If this map is necessarily injective somewhere on the boundary of F, then after a generic perturbation of the α and β curves the moduli space $\hat{\mathcal{M}}(\phi)$ has the expected dimn ($J(t)$ achieves transversality for this domain).

Why? In this case by varying the α_i one can see directly that u lies in a smooth parametrized moduli space.

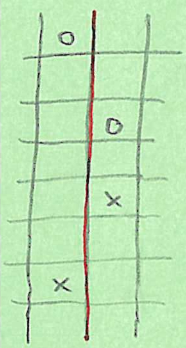
Sarkar's Argument Suppose every region in $\Sigma - \alpha - \beta$ is a square or a rectangle. Then the regions of the surface that have Maslov index one (could possibly be the shadow of something counted by the differential) look like



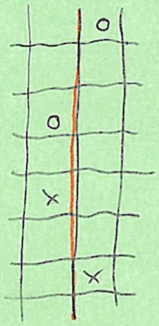
In either case the map $\hat{u}: F \to \Sigma$ must be an embedding, and is in particular boundary injective.

Commutation Invariance: An Example

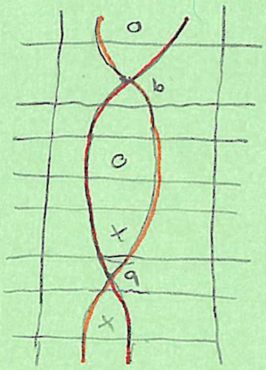
G



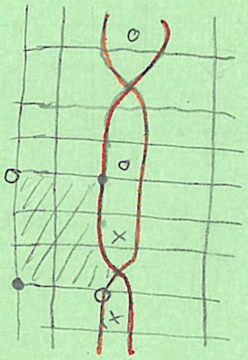
G'



Joint Diagram



There is a chain map $\hat{P}$ from $\widehat{CFK}(G)$ to $\widehat{CFK}(G')$ via counting pentagons which include a.



$\bullet\ \vec{x}$

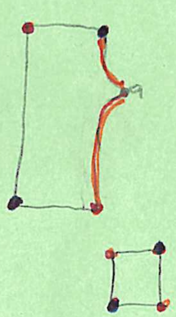
$\circ\ \vec{y}'$

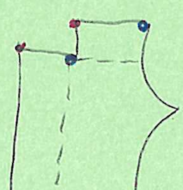
$$P(\vec{x}) = \sum_{\vec{y}' \in S(G')} \sum_{\substack{p \in Pent^{\circ}(\vec{x}, \vec{y}') \\ p \in \mathbb{X} = \phi \\ p \in \mathbb{O} = \phi}} \vec{y}'$$

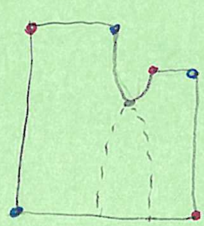
Things to check

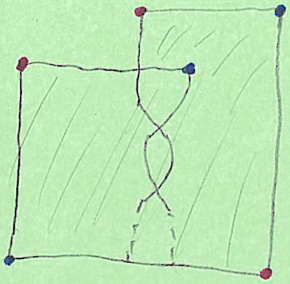
This is a chain map. $P \circ \hat{\partial}_G + \hat{\partial}_{G'} \circ P = 0$

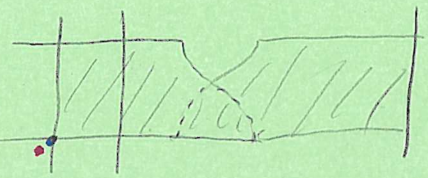
Cases











Exercise: This has six possible decompositions.

One also produces a map P' in the reverse direction...
To verify this is a chain homotopy equivalence, one has to ~~[illegible]~~ produce the homotopy and study its degenerations. It counts hexagons...

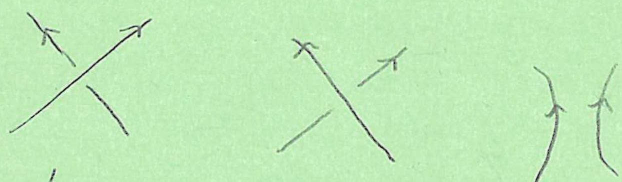
Example:

Some applications

- Explicit computation of $\widehat{HFK}$ for knots of up to twelve crossings [Baldwin - Gillam] https://www.math.uic.edu/~culler/gridlink
- Nice direct proofs of some structural things
    - Skein exact triangle
    - Dbcs
    - $\hat{\tau}$ is a concordance invt & lower bound on the slice genus.

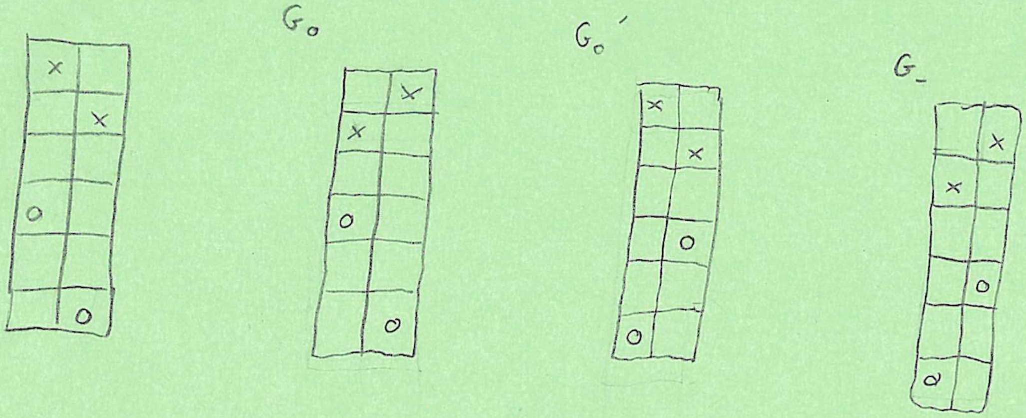
Skein Exact Triangle

$\widehat{HFK}$, HFK^- have skein exact triangles, which are provable directly from grid diagrams.



L_+ L_- L_0

G_+ G_0 G_0' G_-



Statement Recall $\Delta_{L_+}(t) - \Delta_{L_-}(t) = (t^{1/2} - t^{-1/2})\Delta_{L_0}(t)$.

If $l_0 = l+1$ $\exists$ an exact sequence

$$\cdots \to \underset{\text{collapsed Alexander grading}}{\widehat{HFK}_m(L_+, i)} \to \widehat{HFK}_m(L_-, i) \to \widehat{HFK}_{m-1}(L_0, i) \to \cdots$$

If $l_0 = l-1$, $\exists$ an exact sequence

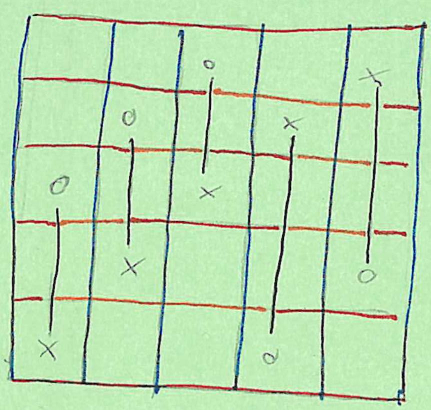
$$\cdots \to \widehat{HFK}_m(L_+, i) \to \widehat{HFK}_m(L_-, i) \to (\widehat{HFK}(L_0) \otimes J)_{m-1,i} \to \widehat{HFK}_{m-1}(L_+, i) \to \cdots$$

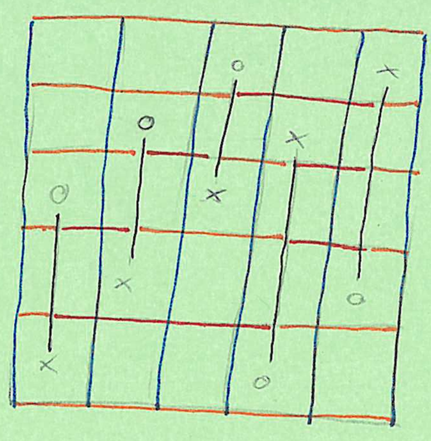
$(0,-1)$

$(1,0)$ $\quad$ $(1,0)$

$(2,1)$

<u>Double branched covers</u>





[Levine "Computing knot Floer homology in cyclic branched covers"]

Extending to the minus case

- One now counts rectangles that go over the O basepoints.
- In the differential one typically counts these separately

  $\mathbb{F}[V_1, \ldots, V_n]$

- In homology all V's on a single component become homologous

  $\rightsquigarrow \mathbb{F}[U_1, \ldots, U_\ell]$

- Can then collapse to $\mathbb{F}[U]$.

- This computes $HFK^-(S^3, K)$, sans extra factors.

The τ-invariant

We can interpret τ as the minimal Alexander grading st the map $\widehat{HFK}(S^3, K, j) \longrightarrow \widehat{HF}(S^3)$ is nontrivial.

To get something that sees this in real time, we note that

$$CFK^-(\mathbb{G})/U_1 \simeq \widehat{CFK}(S^3, K).$$

$\tau(K)$ is the minimal grading that survives in the map between homologies.

<u>Propn</u> $|\tau(K)| \leq g_4(K)$

<u>Corollary</u> $\tau(K) = \frac{(p-1)(q-1)}{2}$ for torus knots, so $u(K) = \frac{(p-1)(q-1)}{2}$ since this can be achieved in practice.