Nice and grid diagrams

Multipointed Heegaard diagrams

We could pick Morse Fts ws w/ multiple index 0 & index 3 critical pts.

\[ H = (\Sigma, \alpha, \beta, z) \text{ or } H = (\Sigma, \alpha, \beta, z, \tilde{w}) \]

\[ z = (z_1, ..., z_n) \]

\[ \tilde{w} = (w_1, ..., w_n) \]

\[ \alpha = (\alpha_1, ..., \alpha_{3n-1}) \]

\[ \beta = (\beta_1, ..., \beta_{3n-1}) \]

We work in Sym^{3n-1}(\Sigma).

Example \((S^3, 3, 1)\)

Why? This is essentially counting \( HF(Y \# (n-1)(S^1 \times S^2)) \). So we end up tensoring w/ a copy of \( V^{3n-1} \) where \( V = \mathbb{R}_x \oplus \mathbb{R}_y \) w/ gradings 0 and -1.
Gradings on $V$ are $(9,0)$, $(-1,1)$.

We might also need multiple basepoints for links.

**Example** $(S^3, (5,9))$

There are two ways to deal with this. One is to attach handles until you have a knot. The other is to keep track of multiple Alexander gradings, one for each component.

$F_2 = \{0, \left(\frac{4}{9}, \frac{2}{9}\right)\}$

$F_2 = \{-1, \left(\frac{4}{9}, \frac{2}{9}\right)\}$

$F_2 = \{-2, \left(-\frac{1}{9}, \frac{2}{9}\right)\}$

$HF^-$ can be over $\mathbb{F}[u]$ or $\mathbb{F}[u, v_2]$.

Note: $X_0(HFL(S^3, L)) = \prod_{i=0}^{\infty} \left(\frac{u}{c_i} \wedge v_2 \wedge \cdots \wedge v_{2k}\right) \Delta_c(t_{1,\ldots, t_{2k}})$

There is a spectral sequence from $HFL(S^3, L)$ to any sublink.
These things can be combined.

Exercise The homology is the previous homology tensored with $V$.

This is background for the next thing.

**Def** (Sarkar–Wang) We say a Heegaard diagram is **nice** if every region of $\Sigma - \partial$ not containing a basepoint is a bigon or a square.

**Thm** (Sarkar–Wang) Suppose $\mathcal{H}$ is a nice diagram. Then any region consisting of an empty bigon or rectangle has a unique pseudoholomorphic representative, and no other region does.

Includes

There is an algorithm for turning any diagram into a nice diagram, but it is long and messy.
Most prominent application: grid diagrams.

Idea Any $L \subseteq S^3$ has a multi-pointed Heegaard diagram on the torus whose curves are arranged as

\[ \begin{array}{c}
\includegraphics[width=0.5\textwidth]{grid_diagram.png}
\end{array} \]

**Defn** A **grid diagram** is an non square marked w/ decorations $x_1, \ldots, x_n; a_1, \ldots, a_n$ s.t.

1. Each row and each column contain exactly one of each type of marking.
2. No square is marked twice.

One can reconstruct a link from a grid diagram via the procedure above.

**Exercise** Any link has a grid diagram

- What links can you make on a $3 \times 3$, $4 \times 4$, $5 \times 5$ grid?

Grid diagrams represent the same link, if they are related by 
Crownwell moves

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\includegraphics[width=0.5\textwidth]{grid_moves.png}
\end{array} \]
Commutation

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\[3\]  Stabilization

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and the other seven options.

Exercise

How do you take a grid diagram for $k$ to a grid diagram for the mirror? How do you take a grid diagram for $k$ to the reverse?

The Sarkar-Wang thm implies that $\text{HF}_k$ can be computed combinatorially from a grid diagram (indeed, you can check invariance etc. this way, and also compare $\text{HF}_k$ and variants). This is of course multipointed.

- Generators are the $n!$ sets of intersection pts, called states, correspond to elements of $S_n$

\[
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(123) \\
(213) \\
(132) \\
(321) \\
(231)
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\]
Some terminology

The set of states is $S(\mathcal{G})$.

The set of rectangles between two states $x$ and $y$ is $\text{Rect}(x,y)$; the set of rectangles from $x$ to $y$ is $\text{Rect}^+(x,y)$. [Not containing a basepoint or another point in the grid state.]

$$\tilde{\mathcal{D}} = \sum_x \sum_y \mathbb{1}_{y \in \text{Rect}^+(x,y)}$$

Gradings

The grading function is determined by

$$M_\emptyset(x,\emptyset) = 0$$

Exercise: Why is this the correct generator to fix?

$$M_\emptyset(x) - M_\emptyset(y) = 1 - 2\pi(r \cap x) + 2\pi(x \cap \text{Int}(D))$$

The Alexander grading for a knot is determined by

$$A(\emptyset,\emptyset) = 0; \quad A(x) - A(y) = \pi(r \cap x) - \pi(r \cap D)$$

or

$$A(x) = \frac{1}{2} \left( M_\emptyset(x) - M_\emptyset(x) \right) - \frac{n-1}{2}$$

This can be written down as an explicit non-relative function.
In consequence, HF\(_K\) can be computed by program, albeit slowly.

Balasubramanian: Prime knots up to eleven crossings.

https://www.math.uic.edu/~culler/gridlink

Extending to the minus case

- One counts rectangles that go over the 0 basepoints.
- In the differential, one counts these separately.

\[ \text{IF} \{V_0, \ldots, V_n\} \]

In homology, all of the Vs on a single component become homologous \[ \text{IF} \{V_0, \ldots, V_n\} \]

can collapse to \[ \text{IF} \{V\} \]

\[ \text{Note} \quad \frac{HF\_K}{HF\_K} \text{ has a skein exact triangle, which can be proved directly from the grid diagrams} \]

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Statement: Recall \( \Delta_{e^+}(k) - \Delta_{e^-}(k) = (e^{\frac{k}{2}} - e^{-\frac{k}{2}}) \Delta_{e^0}(k) \)

If \( \ell_0 = \ell + 1 \) \( \in \) an exact sequence

\[
\ldots \rightarrow \overrightarrow{HFK}(L_\ell, i) \rightarrow \overrightarrow{HFK}(L_{\ell - 1}, j) \rightarrow \overrightarrow{HFK}_{m - 1}(L_{\ell - i}, i) \rightarrow \ldots
\]

If \( \ell_0 = \ell - 1 \) \( \in \) an exact sequence

\[
\ldots \rightarrow \overrightarrow{HFK}(L_\ell, i) \rightarrow \overrightarrow{HFK}(L_{\ell - 1}, j) \rightarrow (\overrightarrow{HFK}(L_{\ell - 1}) \otimes J) \rightarrow \overrightarrow{HFK}(L_{\ell - 2}) \rightarrow \ldots
\]

\[
(0, 1)
\]

\[
(1, 0)
\]

\[
(2, 1)
\]