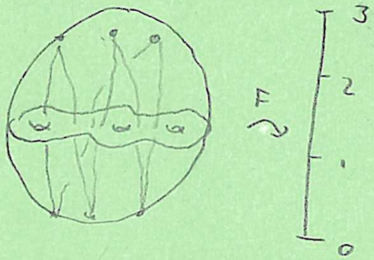
Nice and grid diagrams

MOS
"A combinato

Multipointed Heegaard diagrams

- We could pick Morse ftns w/ multiple index 0 & index 3 critical pts.

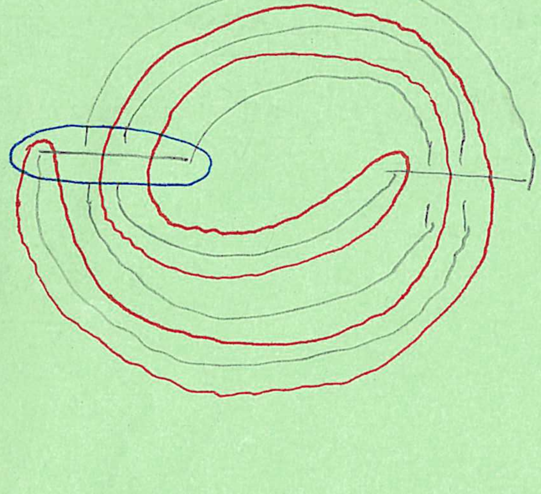


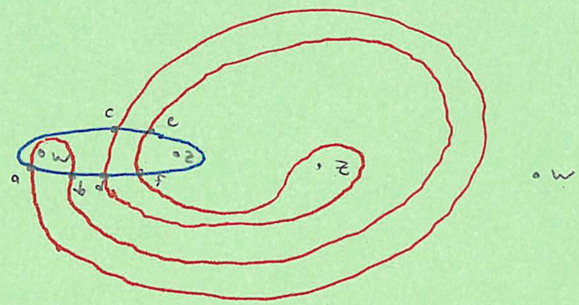
$\mathcal{H} = (\Sigma, \vec{\alpha}, \vec{\beta}, \vec{z})$ or $\mathcal{H} = (\Sigma, \vec{\alpha}, \vec{\beta}, \vec{z}, \vec{w})$

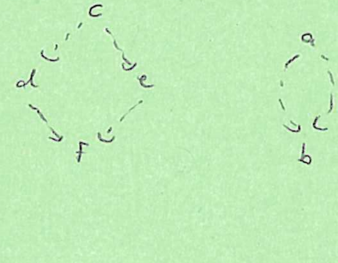
- $\vec{z} = (z_1, \dots, z_n)$
- $\vec{w} = (w_1, \dots, w_n)$
- $\vec{\alpha} = (\alpha_1, \dots, \alpha_{g+n-1})$
- $\vec{\beta} = (\beta_1, \dots, \beta_{g+n-1})$

We work in $\mathrm{Sym}^{g+n-1}(\Sigma)$.

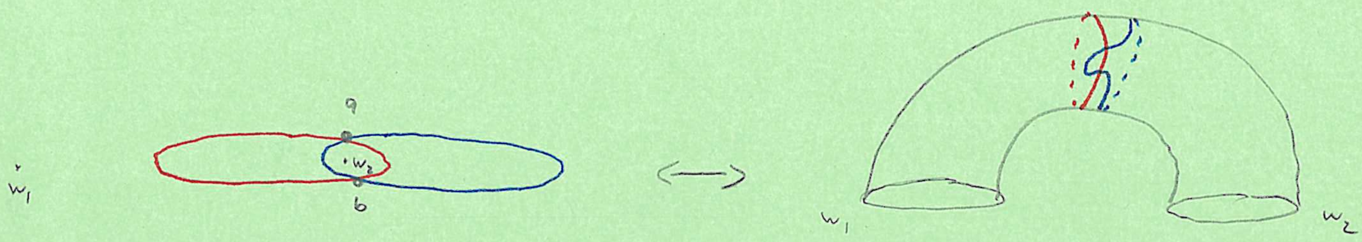
Example $(S^3, 3_1)$

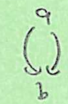


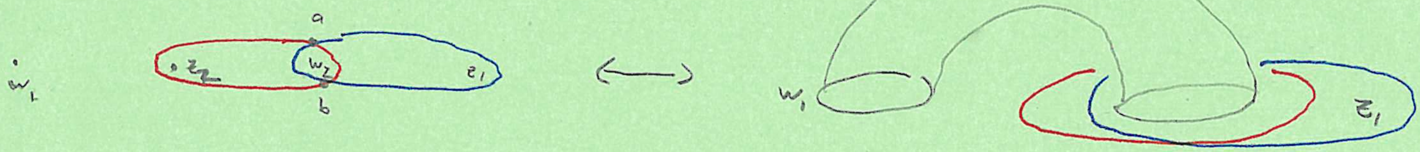


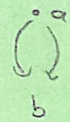


Why? This is essentially counting $\widehat{HF}(Y \#_{(n-1)} (S^1 \times S^2))$. So we end up tensoring w/ a copy of $V^{\otimes(n-1)}$ where $V = \mathbb{F}_2 \oplus \mathbb{F}_2$ w/ gradings 0 and -1.









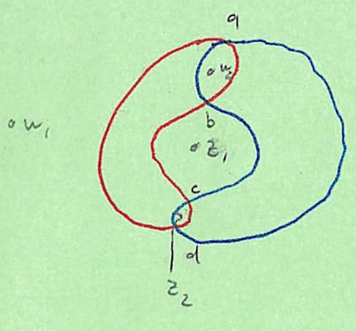
Gradings on V are $(0,0)$.
$(-1,-1)$

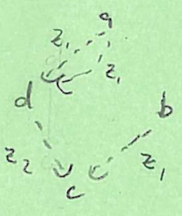
In the minus case things are more delicate and this in fact computes the correct theory without any v's.

We might also need multiple basepoints for links.

Example $(S^3, \bigcirc\!\bigcirc)$

There are two ways to deal with this. One is to attach handles until you have a knot. The other is to keep track of multiple Alexander gradings, one for each component.





$\mathbb{F}_2 \quad (0, (\tfrac{1}{2}, \tfrac{1}{2}))$

$\mathbb{F}_2 \quad (-1, (\tfrac{-1}{2}, \tfrac{1}{2})) \quad \mathbb{F}_2 \quad (-1, (\tfrac{1}{2}, \tfrac{-1}{2}))$

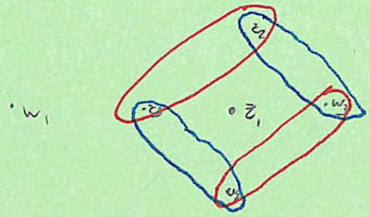
$\mathbb{F}_2 \quad (-2, (\tfrac{-1}{2}, \tfrac{-1}{2}))$

HFK^- can be over $\mathbb{F}[U]$ or $\mathbb{F}[U_1, U_2]$.

<u>Note</u> $\chi(\widehat{HFL}(S^3, L)) \simeq \prod_{i=1}^{\ell} (t_i^{1/2} - t_i^{-1/2}) \, \Delta_L(t_1, \dots, t_\ell)$

There is a spectral sequence from $\widehat{HFL}(S^3, L)$ to any sublink.

These things can be combined.



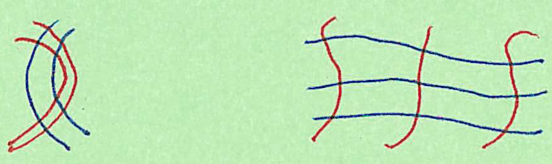
<u>Exercise</u> The homology is the previous homology tensored with V.

This is background for the next thing.

<u>Defn</u> (Sarkar-Wang) We say a Heegard diagram is <u>nice</u> if every region of $\Sigma - \alpha - \beta$ not containing a basepoint is a bigon or a square.

<u>Thm</u> (Sarkar-Wang) Suppose $\mathcal{H}$ is a nice diagram. Then $\exists$ a cpx structure st any region consisting of an empty bigon or rectangle has a unique pseudholo'c representative, and no other region does.

<u>Includes</u>



There is an algorithm for turning any diagram into a nice diagram, but it is long and messy.

Most prominent application: grid diagrams.

Idea Any $L \subseteq S^3$ has a multipointed Heegaard diagram on the torus whose curves are arranged as

DeFn A grid diagram is an $n \times n$ square marked w/ decorations $X_1, \dots, X_n$; $O_1, \dots, O_n$ st

① Each row and each column contain exactly one of each type of marking.

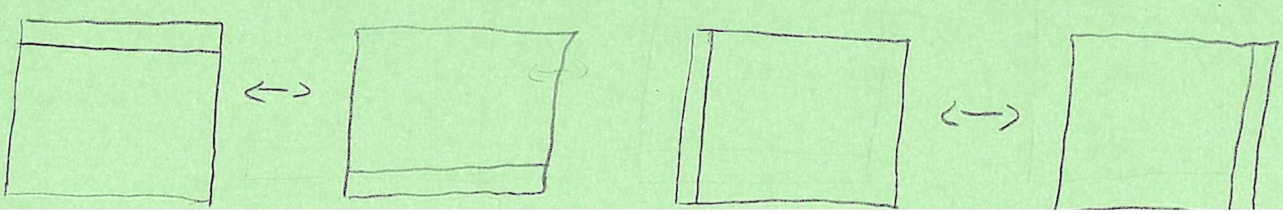
② No square is marked twice.

One can reconstruct a link from a grid diagram via the procedure above.

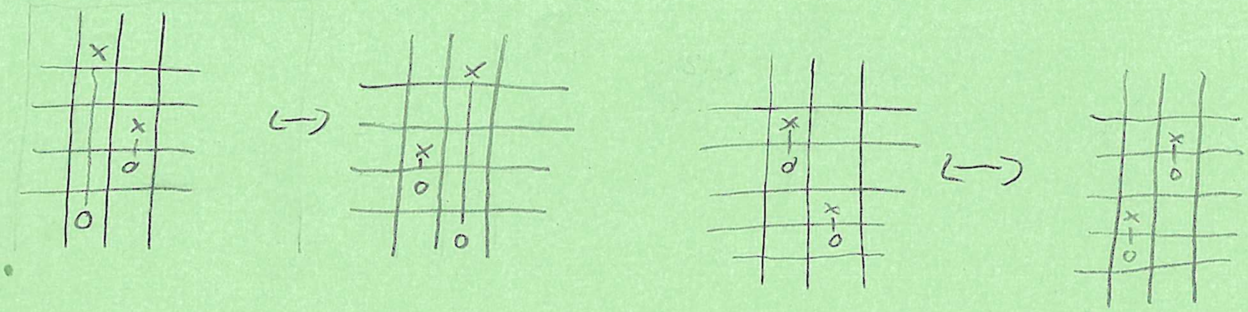
Exercise - Any link has a grid diagram
- What links can you make on a 3×3, 4×4, 5×5 grid?

Grid diagrams represent the same link iF they are related by Crommwell moves

① Translation



2) Commutation

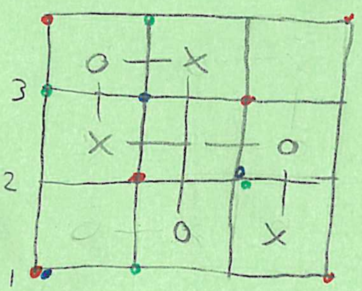


③ Stabilization

and the other seven options.

Exercise: How do you take a grid diagram for K to a grid diagram for the mirror? How do you take a grid diagram for K to the reverse?

The Sarkar-Wang thm implies that $\widehat{HFK}$ can be computed combinatorially from a grid diagram (indeed, you can check invariance etc this way, and also compute HFK^- and variants). This is of course multipointed.



- Generators are the $n!$ sets of intersection pts, called states. Correspond to elements of S_n

(123)

↙ ↓ ↘

(213) (132) (321)

(312)

(231)

Some terminology

- The set of states is $S(G)$.
- The set of rectangles between two states $\vec{x}$ and $\vec{y}$ is $Rect(\vec{x},\vec{y})$; the set of empty rectangles from $\vec{x}$ to $\vec{y}$ is $Rect^\circ(\vec{x},\vec{y})$. [Not containing a basepoint or another point in the grid state.]
- $$\hat{\partial}\vec{x} = \sum_{\vec{y}} \sum_{r \in Rect^\circ(\vec{x},\vec{y})} y$$

Gradings

- The grading function is determined by

$$M_{\mathbb{O}}(x^{NWO}) = 0$$ { Exercise: Why is this the correct generator to fix?

$$M_{\mathbb{O}}(x) - M_{\mathbb{O}}(y) = 1 - 2\#(r \cap \mathbb{O}) + 2\#(\vec{x} \cap Int(r))$$

- The Alexander grading for a knot is determined by

$$A(x^{NWO}) = \tau; \quad A(\vec{x}) - A(\vec{y}) = \#(r \cap \mathbb{X}) - \#(r \cap \mathbb{O})$$

or $$A(\vec{x}) = \frac{1}{2}\left(M_{\mathbb{O}}(x) - M_{\mathbb{X}}(x)\right) - \frac{n-1}{2}.$$

This can be written down as an explicit non-relative function.

In consequence $\widehat{HFK}$ can be computed by program, albeit slowly.

Baldwin - Gillam: Prime knots up to eleven crossings.

https://www.math.uic.edu/~culler/gridlink

Extending to the minus case

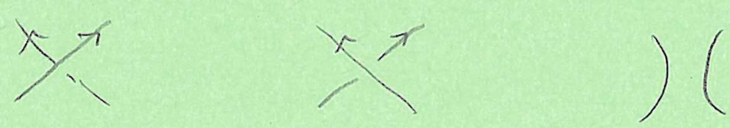
- One counts rectangles that go over the O basepoints.
- In the differential one counts these sepalately

  $\rightsquigarrow \mathbb{F}[V_1, \dots, V_n]$

  In homology, all of the V's on a single component become homologous $\rightsquigarrow \mathbb{F}[U_1, \dots, U_\ell]$.

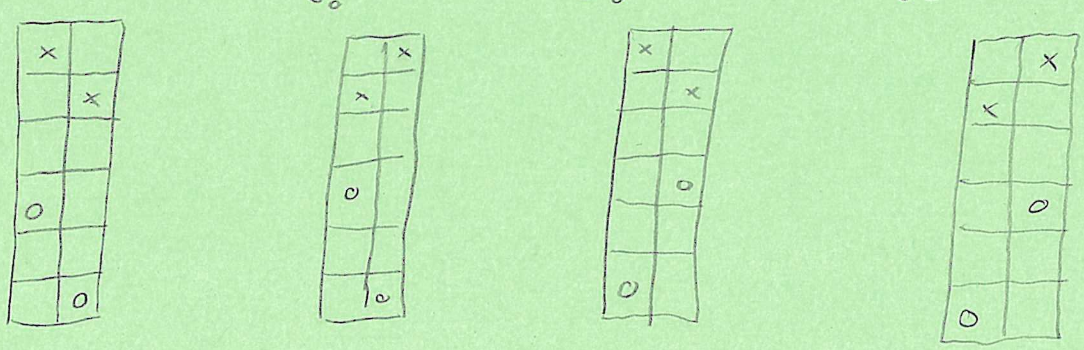
  Can collapse to $\mathbb{F}[U]$.

Note $\widehat{HFK}$ / HFK^- has a skein exact triangle, which can be proved directly from the grid diagrams



L_+ L_- L_0

G_+ G_0 G_0' G_-



(Statement: Recall $\Delta_{L_+}(t) - \Delta_{L_-}(t) = (t^{1/2} - t^{-1/2})\Delta_{L_0}(t)$

If $\ell_0 = \ell + 1$ $\exists$ an exact sequence

$$\cdots \longrightarrow \widehat{HFK}(L_+, i) \longrightarrow \widehat{HFK}(L_-, i) \longrightarrow \widehat{HFK}_{m-1}(L_0, i) \longrightarrow$$

If $\ell_0 = \ell - 1$, $\exists$ an exact sequence

$$\cdots \longrightarrow \widehat{HFK}(L_+, i) \longrightarrow \widehat{HFK}(L_-, i) \longrightarrow \left(\widehat{HFK}(L_0) \otimes J\right) \longrightarrow \widehat{HFK}(L_+, i) \longrightarrow \cdots$$

$(0,-1)$

$(1,0) \quad (1,0)$

$(2,1)$