Lecture 14

Last time

(M, L_0, L_1) symplectic w/ two Lagns. $[\omega]\cdot\pi_2(M)=0$, $[\omega]\cdot\pi_2(M,L)=0$

Could be

- $\pi_2(M) = \pi_2(M, L_i) = 0$

- M exact, convex at ∞, L_i exact

Arnol'd

① If $\varphi : (M,\omega) \to (M,\omega)$ Hamiltonian symplectomorphism w/ isolated fixed pts,

$\#\mathrm{Fix}(\varphi) \geq \sum b_i(M)$.

Arnol'd - Givental

② If $\varphi : (M,\omega) \to (M,\omega)$ Hamiltonian symplectomorphism w/ $L \pitchfork \varphi(L)$

then $\#(L \pitchfork \varphi(L)) \geq \sum b_i(L)$.

Say $\pi_2(M) = 0$. Then $\pi_2(M^- \times M, \Delta) = 0$. Given $\varphi : M \to M$ Hamiltonian. $1\times\varphi : M\times M^- \to M\times M^-$ is Hamiltonian. If φ has isolated fixed pts, then $\Delta \pitchfork 1\times\varphi(\Delta) = \Delta \pitchfork \Gamma_\varphi$. So if we know Arnol'd Givental for $\pi_2(M,L)=0$, we know

$$\#\mathrm{Fix}(\varphi) = \Delta \pitchfork \Gamma_\varphi \geq rk(HF(\Delta, \Gamma_\varphi)) \geq \sum_i b_i(\Delta) = \sum_i b_i(M).$$

Proving Arnol'd - Givental

Weinstein Tubular Nbhd Thm Let (M,ω) be symplectic, L cpt Lagn, ω_0 the canonical symplectic form on T^*L, $i_0 : L \hookrightarrow T^*L$ Lagn embedding, $i : L \hookrightarrow M$. Then there are nbhds U_0, U of L in T^*L, M and a symplectomorphism ψ st

$$\begin{tikzcd} U_0 \arrow[rr, "\psi"] & & U_1 \\ & L \arrow[ul, "\iota_0"] \arrow[ur, "\iota"'] & \end{tikzcd}$$

So it suffices to prove the claim for $L \subseteq T^*L$.

Secondly, everything about this setup is invariant under Hamiltonian isotopy (one can define the differential as associated to a pair (J, H), where the first thing one does is apply φ_{H_1} to L_1).

So it doesn't matter <u>which</u> Hamiltonian we use.

- In T^*L, let L_0 be the 0-section and L_1 be the graph of the one-form df for f Morse, so $L_0 \pitchfork L_1 = \mathrm{Crit}(f)$.
  - <u>Exercise</u> $df = \varphi(L_0)$ for φ a suitable Hamiltonian isotopy.

- $CF(L_0, L_1) = C_*^{\text{Morse}}$

- $HF(L_0, L_1) \cong H_*(L)$.] one checks that the pseudoholo curves correspond precisely to the Morse flowlines.

Extending to the HF case - salient issues

- Making Sym^g symplectic
- Exactness issues
- Gradings - need to understand the index better.

$Sym^g(\Sigma)$

- Picking a complex structure on Σ gives a symplectic area form wrt the α_i, β_i are Lagn.

- If we puncture the surface at z, this can be made convex at ∞ w/ the α_i, β_i exact Lagns.

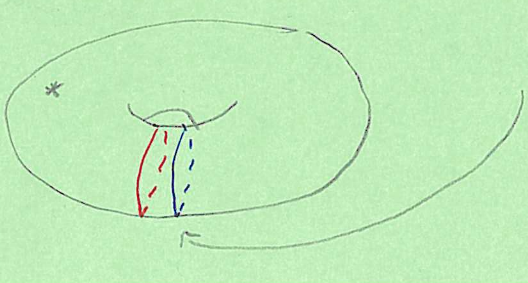
$$F : \Sigma \longrightarrow \mathbb{R}$$

- There is an (a priori <u>singular</u>) exhausting function on $Sym^g(\Sigma)$ descending from $F \times \dots \times F$ on Σ^g. Near the Lagns, F is smooth and $-d \circ J \circ dF \simeq \omega^n$. So eg $\Pi\alpha$, $\Pi\beta$ still exact Lagns.

<u>Perutz</u> F can be smoothed to a plurisubharmonic exhausting function on $Sym^g(\Sigma - w)$. [More generally, on $Sym^g(\Sigma)$, $\omega^{\wedge n}$ can be smoothed to a form which agrees w/ ω^n near the Lagns]

This possibly requires isotopies of the Lagns.

<u>Also Perutz</u> Isotopies (easy) and handleslides (hard) can be expressed as Hamiltonian isotopies of the Lagns w/ invariance.

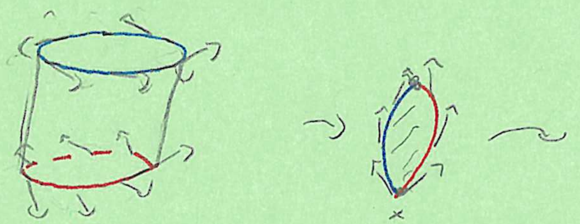


We cannot make both of these exact at once.

Index and Finiteness Issues

What is a Maslov index?

First suppose we have a disk from x to itself (aperiodic domain).



(M, L_0, L_1)

Compare (T^*M, T^*L_0) and (T^*M, T^*L_1)

$\cong$

$(\mathbb{C}^n, \mathbb{R}^n)$

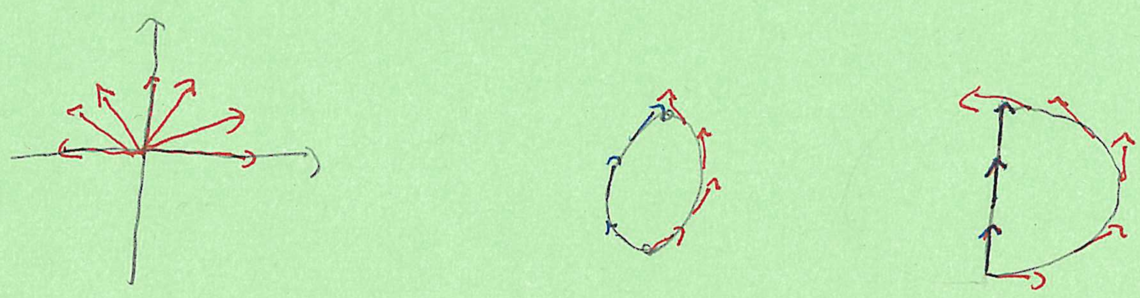
The Maslov index $\mu(M) = 2c_1\big(T^*M \times [0,1], (T^*L_0 \otimes_{\mathbb{R}} \mathbb{C} \times \{0\}) \oplus (T^*L_1 \otimes_{\mathbb{R}} \mathbb{C} \times \{1\})\big)$

$U(n)/O(n)$ The map is via degree of the determinant map.

$\cong$

Another phrasing: $\pi_1(Gr_L(\mathbb{C}^n)) \cong \mathbb{Z}$. There is a map $\mu : \pi_1(Gr_L(\mathbb{C}^n)) \to \mathbb{Z}$ which is essentially a winding number.

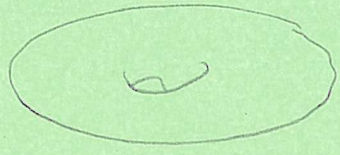
In $\mathbb{C}$, "1" is the path:



Note this means a path from x to itself will always return an even number.

What could cause problems here?

- The presence of two disks having different Maslov index between two points
- This implies there is a disk of nonzero Maslov index from x to itself. So there certainly isn't a relative $\mathbb{Z}$-grading.



A disk from x to itself is of course a periodic domain. Recall that $\pi_2(x,x) \simeq H^2(X)$

Exercise This depends on the Chern class of the spin^c-structure. We can in general grade by $\mathbb{Z}/_{c_1(s)}$.

What else?

- If we don't puncture Σ, $Sym^g(\Sigma)$ may have some π_2.
  - $g>2 \rightsquigarrow \pi_2(Sym^g(\Sigma)) \simeq \mathbb{Z}$
  - $g=2 \rightsquigarrow \pi_2'(Sym^g(\Sigma)) \simeq \mathbb{Z}$
- Solution: Count intersections w/ $\{z\} \times Sym^{g-1}(\Sigma)$, and work over $\mathbb{F}[U]$.