Lecture 13

References
- Hutchings = Morse Theory w/ an.
  eye toward pseudo-hols
- Pedroza = A quick view of Lagn
  Floer homology

- McDuff–Salamon = J-hol'c curves
  ^3 symplecti
  topology

Last time

Q what is the minimum number of fixed pts of a Hamiltonian symplectomorphism $\ell: (M, \omega) \longrightarrow (M, \omega)$?

Arnol'd Conjectures

① Let M cpt symplectic, and $\ell: M \to M$ a Hamiltonian symplectomorphism w/ nondegenerate critical pts. Then

$$|Fix(\ell)| \geq \sum_i b_i(M).$$

② (Arnol'd – Givental) Let L be a Lagn submfd of (M, ω) and ℓ a Hamiltonian diffeomorphism of M^{2n} st $L \pitchfork \ell(L)$. Then $|L \pitchfork \ell(L)| \geq \sum_i b_i(L)$.

Inspired by Poincaré–Birkhoff An area-preserving diffeo of the annulus, which maps the bdy circles to themselves in the opposite directions has at least two fixed pts.

Floer, 1980s
  ⓐ ① True for $\pi_2(M) = 0$; ② true for $\pi_2(M) = 0$, $\pi_2(M, L_i) = 0$.
  ⓑ (slightly different methods) Also true if $[\omega] = a \cdot c_1(M)$, $a > 0$.

Other people $\leadsto$ more general cases.

Tool

Lagn Floer (co)homology $(M, L_0, L_1) \leadsto HF(L_0, L_1)$

$\underbrace{\qquad\qquad}$

graded (by something, $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}$ today)
vector space (over something,
today $\mathbb{F}_2$)
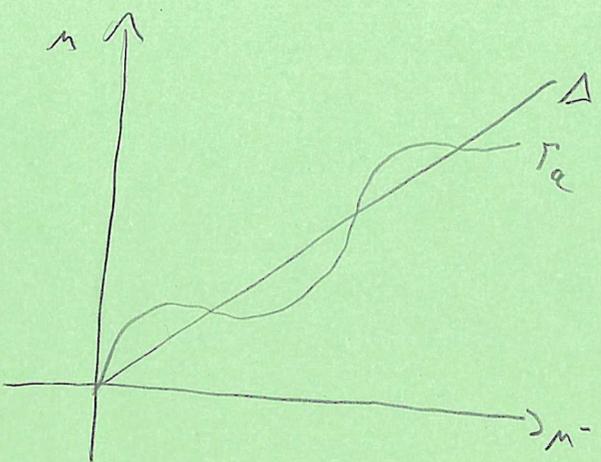
First we need the notion of an acs.

<u>Defn</u> $J: TM \to TM$ is an almost complex structure if $J: TM_x \to TM_x$ and $J^2 = -1$.

- Not every symplectic mfd admits a cpx structure, but every symplectic mfd admits an acs. [eg $\mathbb{CP}^2 \#_9 \overline{\mathbb{CP}^2}$ Fintushel-Stern, Park]

- J is ω-compatible if $\omega(v, Jv) > 0$ when $v \neq 0$ and $\omega(Jv, Jw) = \omega(v, w)$.

- If J is ω-cptble, J determines a metric via $g(v, w) = \omega(v, Jw)$, [Indeed, any two of (ω, J, g) determine the third.]

<u>Exercise</u> The space of J compatible w/ ω is contractible.

<u>Assumptions on (M, L_0, L_1)</u>

- $[\omega] \cdot \pi_2(M) = 0,$ $[\omega] \cdot \pi_2(M, L_i) = 0;$ L_i cpt, $L_0 \pitchfork L_1.$ (Easy to remove

  <u>How could you get this?</u>
  - Could just have both groups 0.
  - Could also have M exact w/ exact Lagrangians. (why?)
  - One generally wants M to be <u>convex at infinity</u>, in this case; $\exists$ a function $f: M \to \mathbb{R}$ proper, bdd below s.t. $\omega = -d\circ J \circ df.$

'F is an exhausting function or a radial function or a strictly plurisubharmonic ftn.

$\underline{Examples}$ ① $F: \mathbb{C} \longrightarrow \mathbb{R}$

$$(x,y) \longrightarrow \tfrac{1}{4}(x^2+y^2)$$

$$dF = \tfrac{1}{2}(x\,dx + y\,dy)$$

$$J_0 \circ dF = \tfrac{1}{2}(-x\,dy + y\,dx)$$

$$-d(J_0 \circ dF) = -\tfrac{1}{2}(-dx \wedge dy + dy \wedge dx) = dx \wedge dy$$

$\underline{Exercise}$ The cotangent bundle of a cpt mfd is convex at ∞.

'Note that these conditions can be weakened in various ways, sometimes at the cost of worse coefficients.

We want to consider pseudoholic maps: (For some family $J(t)$).

$$m(x,y) = \Big\{$$

$\mathbb{R} \times [0,1]$

(s,t)
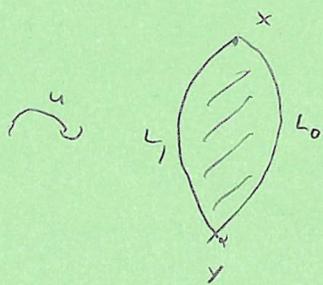
$u: \mathbb{R} \times [0,1] \longrightarrow M$

$u(s,0) \in L_0, \quad u(s,1) \in L_1$

$\partial_s u + J(t,u)(\partial_t u) = 0$

$$\lim_{s \to +\infty} u(s,t) = x$$

$$\lim_{s \to -\infty} u(s,t) = y$$

$J_t \circ du = du \circ j$

We let $CF(L_0, L_1) = \mathbb{F}_2 \langle L_0 \pitchfork L_1 \rangle$, $\partial : CF(L_0, L_1) \circlearrowleft$ counts disks.

<u>Mimics Morse theory</u> We look at the path space $P(L_0, L_1)$.
In the exact w/ exact Lagns case, we have an action
functional $A(\gamma) = \int_0^1 \gamma^*(\lambda) + F_0(\gamma(0)) - F_1(\gamma(1))$, where $\lambda|_{L_i} = dF_i$.
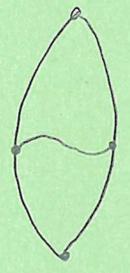
- Critical pts are constant paths
- Flowlines are pseudo-holc curves,
- "Indices" are nastily infinite dimensional.
- For more general cases, one considers a cover of P.

<u>Note</u> $m(x,y) = \coprod\limits_{\substack{\text{homology} \\ \text{classes } \beta}} m^\beta(x,y)$

<u>Want</u> <u>Transversality</u> For a generic choice of $\vec{J}$, $m^\beta(x,y)$ is a smooth mfd of dimn $\mu(\beta) - 1$, where $\mu(\beta)$ is an algebraic-topological additive quantity associated to x, y, and β.

(the set of appropriate J has second category in all ω-cptble J, i.e. contains the intersection of countably-many open dense subsets

<u>Consequence of</u> Sard-Smale Thm, Implicit Ftn Thm for Banach mfds, Riemann-Roch

<u>Compactness</u> There is a cpt space $\bar{m}^\beta(x,y) \supseteq m^\beta(x,y)$ where the points in $\bar{m}^\beta(x,y) \setminus m^\beta(x,y)$ correspond to broken holc disks
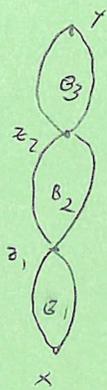
Consequence of "Energy bounds" from ω-compatibility of $\bar{J}$, Sobolev theory.

Gluing For generic $\vec{J}$, $\overline{m}^B(z,y)$ is a (topological) mfd w/ corners, where a k-times broken disk is a codimn-k corner.

i.e. given a k-times broken disk, $(u_1, \ldots, u_{k+1}) \in \overline{m}(x, z_1) \times \ldots$
$$\times \overline{m}(z_k, y)$$

$\exists$ a nbhd homeomorphic to $[0, \infty)^k \times \mathbb{R}^{\mu(B) - k - 1}$.

Consequence of Banach mfd theory, especially Banach Fixed pt thm.

Issues ① why is $\#\overline{m}^B(x,y)$ finite if $\mu(B) = 1$? ⎤
        ② why is $\partial^2 = 0$ ⎟ As in Morse
        ③ why is this well-defined?   ⎟ theory case.
        ④ why is $\sum_B \underline{\qquad}$ finite?] Energy estimates.

$$HF(L_0, L_1) = H_*(CF(L_0, L_1), \partial).$$

What problems are our assumptions avoiding?

Good

Good   Bad   These would have
                    J-holé   positive ω-area,
                    sphere   hence don't arise.
   Disk w/ bdy in L_i
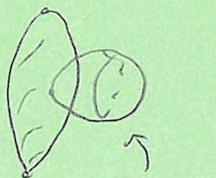
What is compactness at ∞ doing?

· Recall the maximum-modulus principle: A holomorphic
function on a ^{cpt} domain D in $\mathbb{C}$ takes its maximum on the
boundary.

Prevents
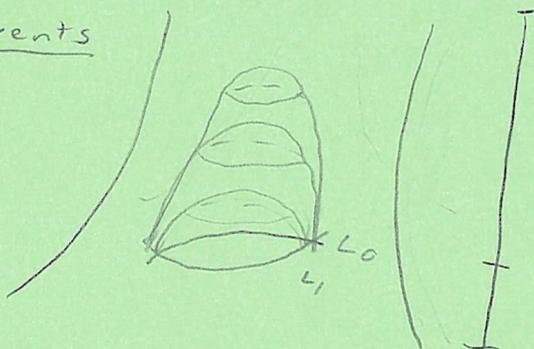
Ensures all disks are
contained in a cpt set.

Invariance One picks an $\mathbb{R}$'s worth of $\vec{J}$'s

Eventually constant
at $\vec{J}_2(t)$

Eventually constant
at $\vec{J}_1(t)$
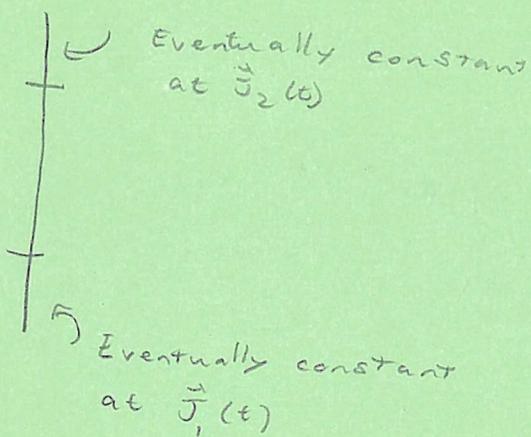
$\bar{\Phi} : CF(L_0, L_1; J_1) \longrightarrow CF(L_0, L_1, J_2)$

$$\bar{\Phi} x = \sum_{\substack{B \\ ind(B) = 0}} {}^{\#} m(x, y) \, y$$

[No need to mod by translation]

Why is this a chain map?

The body of the index-two moduli
spaces breaks as
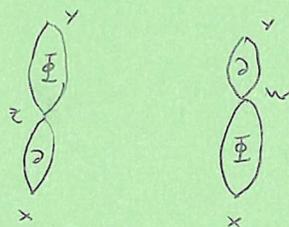
Examples (on surfaces)

① $M = T^* S^1 = \mathbb{R} \times S^1$ $L_0 = \{(0, \theta)\}$ $L_1 = \{(\theta, \sin\theta)\}$
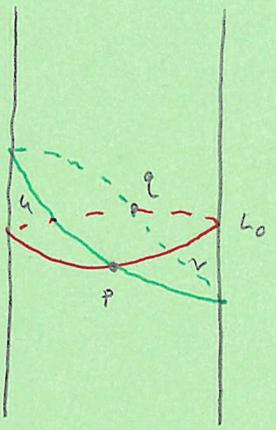
$$CF(L_0, L_1) = \mathbb{F}_2 \langle p \rangle + \mathbb{F}_2 \langle q \rangle$$

$$\partial q = 2p = 0 \qquad HF(L_0, L_1) = \mathbb{F}_2^{\oplus 2}$$

Riemann mapping thm

In order For
L_1 to be exact,
area(u) = area(v)

②

$a \quad c$
$\searrow \swarrow$
b
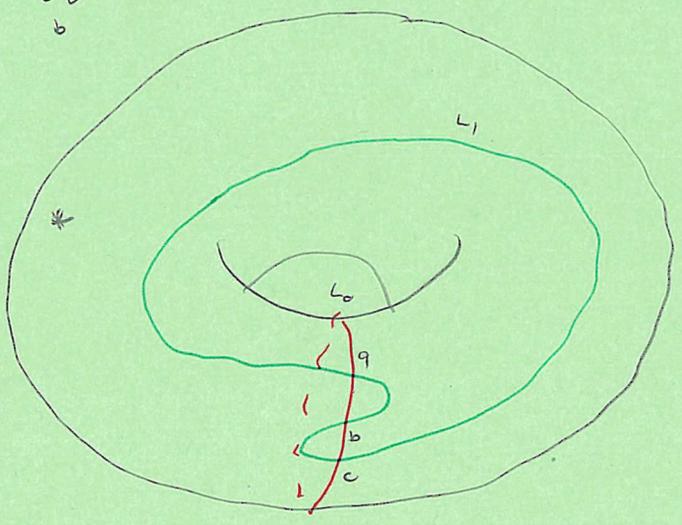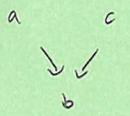
③ But Not
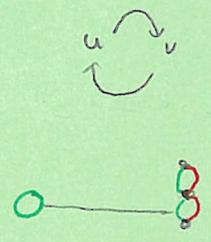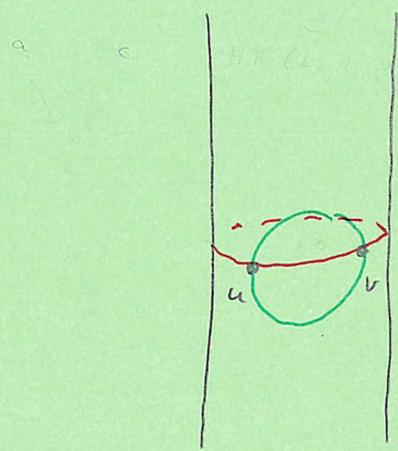
Note This is all invariant under Hamiltonian isotopies, by another application of continuation map style arguments.

Steps in Floer's Proof

• Proved Arnol'd - Givental as follows:

• Let $L \subseteq M$, $\pi_2(M, \omega) = 0$. Then by the Weinstein tubular nbhd thm, we can embed a copy $T^*L \hookrightarrow M$ into M symplectically w/ L as the zero-section. So suffices to work in T^*L.

• In T^*L, let L_0 be the 0-section and L_1 be the graph of the one-form df for f Morse, so $L_0 \pitchfork L_1 = \text{Crit}(f)$.

• $L_1 = \varphi(L_0)$ for φ a Hamiltonian isotopy.

• $CF(L_0, L_1) = C_*^{\text{Morse}}$

• $HF(L_0, L_1) \cong H_*(L)$. But the theory is invt under Hamiltonian isotopy, so this is true for any Hamiltonian φ.

• Now we apply this to $(M \times M^-, \Delta, \Gamma_\varphi)$. We see

$$\# \text{Fix}(\varphi) = \Delta \pitchfork \Gamma_\varphi \geq \text{rk } HF(\Delta, \Gamma_\varphi) \geq \sum_i b_i(\Delta) = \sum_i b_i(M).$$

[Note also $\pi_2(M) = 0 \Rightarrow \pi_2(M \times M^-, \Delta) = 0$].

How does this apply to the HF case?

• Making Sym^g symplectic

• Exactness issues?

• Gradings — need to understand the index slightly better.