An interlude: symplectic geometry background

Start w/ a review of Morse homology.

Recall $F: M \to \mathbb{R}$ is Morse if at any critical point of F,

(M: smooth closed mfd)

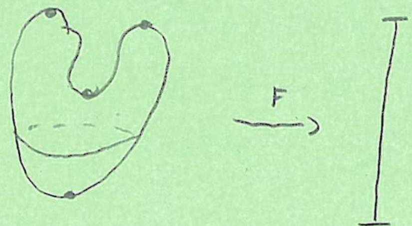
$\mathrm{Hess}_p(F)$ is nonsingular.

- $d_pF = 0$
- Matrix of second derivatives is nonsingular

$\{\text{Morse, Ftns}\}$ are an open dense set in $\{C^\infty \text{ ftns } f: M \to \mathbb{R}\}$

$\mathrm{Crit}(F) = \{p \in M : d_pF = 0\}$. Near $p \in \mathrm{Crit}(F)$, $\exists$ coordinates $x_1, \dots, x_n$ st $x_i(p) = 0$, $F(x_1, \dots, x_n) = -x_1^2 + \dots - x_i^2 + x_{i+1}^2 + \dots + x_n^2$.

The number i is the index of p.



Fix a metric g on M, and consider the vector field ∇F.

Gradient flow lines $\gamma: \mathbb{R} \to M$ st $\gamma'(t) = -\nabla_{\gamma(t)} F$

For $x, y \in \mathrm{Crit}(F)$, let $\widehat{m}(x,y) = \{\text{gradient flow lines } \gamma: \mathbb{R} \to M \text{ st } \lim_{t \to -\infty} \gamma(t) = x,\ \lim_{t \to \infty} \gamma(t) = y\}$

$\exists$ an $\mathbb{R}$-action on $\widehat{m}(x,y)$ by translation. Let $m(x,y) = \widehat{m}(x,y)/\mathbb{R}$.

We define a chain cpx $C_* = C_*^{Morse}(M; \mathbb{F}_2)$ where $C_i = \mathbb{F}_2\langle \{\text{critical pts of index } i\}\rangle$ and $\partial_i : C_i \longrightarrow C_{i-1}$ is given by $\partial_i(x) = \sum_{\substack{y: \\ ind(y) = i-1}} (\# m(x,y)\, y)$. This computes $H_i(M; \mathbb{F}_2)$.

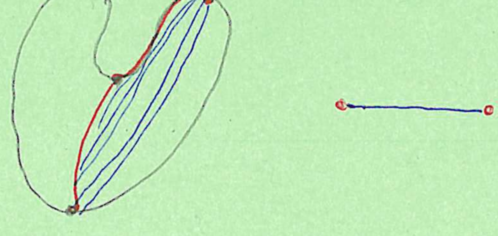
Want
- $\partial^2 = 0$
- $\# m(x,y)$ is a finite number

Uses ① Transversality For a generic choice of Morse Fcn (or metric), the space $m(x,y)$ is smooth of dimn $ind(x) - ind(y) - 1$.

[Fails for ② $\longrightarrow$ [.]

② Compactness Given a sequence of flowlines $\gamma_i \in m(x,y)$ $\exists$ a subsequence $\{\gamma_{i,j}\}$ which converges (in an appropriate sense) to a possibly broken flowline from x to y.

Broken flowline from x to y: A sequence of flows $\eta_1, \ldots, \eta_k$ where η_i is a flow from z_i to z_{i+1}, $z_0 = x$, $z_k = y$.



Let $\overline{m}(x,y) = \{$possibly broken flows from x to $y\}$. Then $\overline{m}(x,y)$ is cpt.

<u>Exercise</u> Formulate a precise notion of the convergence here.

(3) <u>Gluing</u> $\overline{m}(x,y)$ is a manifold w/ corners, for a generic choice of Ftn and metric.

<u>More precisely</u> Near any broken flowline $(\gamma_1, \ldots, \gamma_k)$ from x to y, $\overline{m}(x,y)$ is homeomorphic to $(\mathbb{R}_+)^{k-1} \times \mathbb{R}^{\operatorname{ind}(x) - \operatorname{ind}(y) - k}$.

- We only need $k=2$ today
- Near broken flows, $\overline{m}(x,y)$ looks like $\mathbb{R}_+ \times \mathbb{R}^n$.
  i.e. $\overline{m}(x,y)$ looks like a mfd w/ bdy

<u>Lemma</u> $\partial^2 = 0$

<u>Pf</u> Consider $\partial^2 x$, for $\operatorname{ind}(x) = n$.

$$\partial^2 x = \partial \Big(\sum_{y : \operatorname{ind}(y) = n-1} \# \underbrace{m(x,y)}_{= \overline{m}(x,y)}\, y \Big)$$

$$= \sum_{y : \operatorname{ind}(y) = n-1} \# m(x,y) \cdot \sum_{z : \operatorname{ind}(z) = n-2} \# m(y,z)\, z$$

Coefficient of z in $\partial^2 x$ is $\displaystyle\sum_{y : \operatorname{ind}(y) = n-1} (\# m(x,y))(\# m(y,z))\, z$

$$= \sum_y \#(m(x,y) \times m(y,z))$$

$= \#$ (once-broken flows from x to z)

$= \# \partial \overline{m}(x,z)$

-This is the entire ∂ b/c $\dim \bar{m}(x,z) = 1$, so in the bdy we only get once-broken flows, which are codimn one "corners".

But $\bar{m}(x,z)$ is a cpt one-mfd w/ bdy (①, ③). So

$\# \partial \bar{m}(x,z) \equiv 0 \pmod 2$.

Properties ① and ③ also appear in the defn of ∂:

$$\partial x = \sum_{\substack{y : \mathrm{ind}(y) \\ = \mathrm{ind}(x) - 1}} \# m(x,y)\, y$$

cpt 0-mfd, hence a finite sum

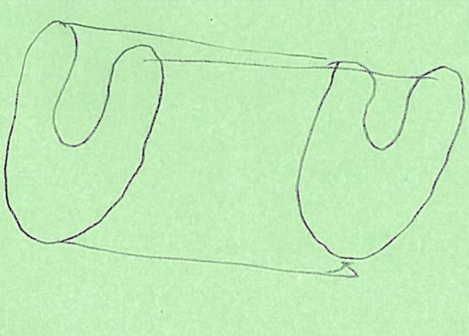
If we work over $\mathbb{Z}$ $\rightsquigarrow$ need orientations for $m(x,y)$ which are "coherent" to get $\partial^2 = 0$.

<u>Exercise</u> ① Formulate what "coherent orientations of moduli spaces" means.

② Show for $M = S^1$ there are two nonequivalent choices of coherent orientations.

<u>Independence of the Morse Ftn</u> Say we have f, g Morse on M.

Pick a ftn $F : M \times [0,1] \to \mathbb{R}$ w/ critical pts only on $M \times \{0\}$ and $M \times \{1\}$ st ∇F looks like ∇f on $[0, \varepsilon)$ and ∇g on $(1-\varepsilon, 1]$



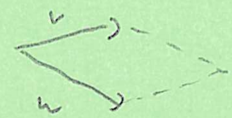
Define a chain map $\Phi : C_*^f \longrightarrow C_*^g$ by $\Phi(x) = \sum_{\substack{\mathrm{ind}(x) \\ = \mathrm{ind}(y)}} \begin{pmatrix} \# \text{ of flows} \\ \text{of } \nabla F \text{ from} \\ (0,x) \text{ to } (1,y) \end{pmatrix} y$.

On to symplectic geometry

Defn A symplectic mfd is a smooth $2n$-dim'l mfd M w/ a 2-form ω st

- ω is closed ($d\omega = 0$)
- ω is nondegenerate (If for $v \in T_pM$, $\omega(v,w) = 0 \ \forall w \in T_pM$ then $v = 0$, or equivalently ω^n is a volume form.)

"A space w/ a notion of area"



Examples ⓪ $(\mathbb{R}^{2n}, \omega_0)$ w/ $\omega_0 = dx_1 \wedge dy_1 + \dots + dx_n \wedge dy_n$

$\mathbb{R}^{2n} = \{(x_1, y_1, \dots, x_n, y_n)\}$

Note that if $n=1$ this is the usual cross product.

In fact, locally this example is everything.

Thm (Darboux) Given $p \in (M^{2n}, \omega)$, there is a nbhd U of p w/ coordinates $(x_1, y_1, \dots, x_n, y_n)$ st $\omega|_U = \omega_0$.

Symplectic topology is therefore exclusively about global behavior.

① Your favorite orientable surface — pick an area form.

②
$$\begin{tikzcd} & S^2 \times S^2 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ S^2 & & S^2 \end{tikzcd}$$

$$\underbrace{\omega = c(\pi_1^* \sigma + \lambda \pi_2^* \sigma)}_{\text{up to scaling, this is everything (Macduff-Salamon}} \qquad \int_{S^2} \sigma = 1$$

(3) $T^4 = (\mathbb{R}^4, \omega_0)/\Gamma$ } Are there others? Unknown.

Γ ← some lattice

(4) $\mathrm{Int}(D^2) \times \mathbb{R}^2$ Not the same as $\mathbb{R}^4$.

x_1, x_2 x_3, x_4

(5) Cotangent bundles (T^*X, ω), X any smooth mfd.

If $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ are local coordinates on $T^*U \simeq U \times \mathbb{R}^n$,

$\omega = -d(\sum y_i\, dx_i)$. [$\omega$ is exact, $\omega = d\lambda$ for a one form λ.

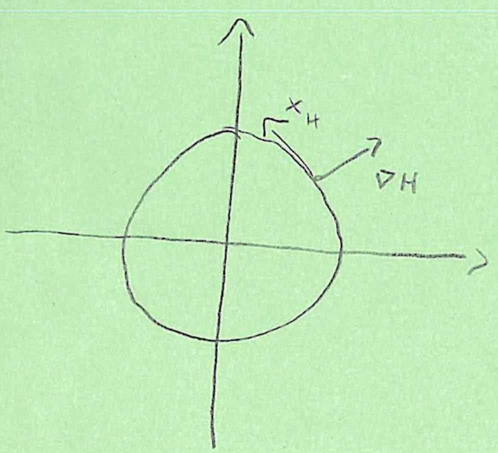
(6) Not S^{2n}, $n > 1$. (why not?)

A symplectomorphism is a map $\varphi : (M, \omega) \longrightarrow (M, \omega)$ st $\varphi^*\omega = \omega$.

There are lots of symplectomorphisms.

Given $H_t : M \longrightarrow \mathbb{R}$ smooth (a time-dependent Hamiltonian), we have a one-form dH_t and a vector field X_{H_t} st $X_{H_t} : \omega(X_{H_t}, \cdot) = dH_t($

Example on $\mathbb{R}^2$, we could have $H(x,y) = \frac{1}{2}(x^2 + y^2)$.

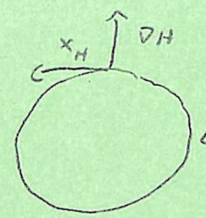


$dH = x\,dx + y\,dy$ $\quad dH_{(x,y)}(v_1, v_2) = xv_1 + yv_2$

$X_H = (y, -x)$

$\omega((y, -x), (v_1, v_2)) = yv_2 + xv$

And in general one has



<u>Exercise</u> $H : (\mathbb{R}^4, \omega) \longrightarrow \mathbb{R}$?

$(x_1, y_1, x_2, y_2) \longmapsto x_1$

We can look at the flow φ_t of X_{H_t}, i.e. the family of diffeomorphisms st $\frac{d}{dt}\varphi_t\big|_x = X_{H_t}(\varphi_t(x))$. These are all symplectomorphisms. A symplectomorphism is <u>Hamiltonian</u> if it is the time-one flow of some H_t.

<u>A classical question</u> What is the smallest number of fixed pts a Hamiltonian symplectomorphism can have?

[Lefschetz Fixed Pt Thm: Bounded below by $\sum (-1)^i b_i(M)$. But we can do better.]

We'll come back to this question in a moment. First, some special submfds of symplectic mfds.

<u>Defn</u> An n-dimensional submfd of (M^{2n}, ω) is <u>Lagrangian</u> if $\omega(v, w) = 0 \;\; \forall \;\; v, w \in T_pL$, $p \in L$.

<u>Exercise</u> This is the highest dimn a submfd $K \subseteq M$ st $\omega|_K = 0$ can possibly have.

Examples ① The plane $\{(x_1, 0, x_2, 0)\}$ inside of $\mathbb{R}^4$.

② Any curve in an orientable surface

③ $(S^1)^n \subseteq \mathbb{R}^{2n}$

④ The 0-section inside of T^*X

(Nearby Lagn conjecture: any $L \subseteq T^*X$ which is closed and exact, i.e. $\omega = d\lambda$ and $d\lambda|_L = dF$, is Hamiltonian isotopic to the zero section. Known for S^1, S^2.)

⑤ The graph and diagonal of a symplectomorphism

- If M symplectic, $M^- \times M$ has a symplectic form $-\omega \oplus \omega$.
- Contains $\Delta = \{(x, x)\}$ the diagonal.
- $\Gamma_\varphi = \{(x, \varphi(x))\}$ the graph of φ.

[Exercise Check these are Lagn. If M exact, φ Hamiltonian, check these are exact Lagns.]

Note $\Delta \cap \Gamma_\varphi$ is exactly the fixed pts of φ.

Arnol'd Conjectures ① Let M cpt symplectic, $\varphi : M \to M$ Hamiltonian symplectomorphism w/ non-degenerate critical pts. Then $|\mathrm{Fix}(\varphi)| \geq \sum_i b_i(M)$.

Arnold-Givental ② Let L be a Lagn submfd of (M, ω), and φ a Hamiltonian symplectomorphism of M^{2n} st $L \pitchfork \varphi(L)$. Then $|L \pitchfork \varphi(L)| \geq \sum_i b_i(L)$.

(Many other versions, eg of # of critical pts of a Morse fcn)

Floer, 1980s (a) (1) True for $\pi_2(M)=0$; (2) true for $\pi_2(M)=0$, $\pi_2(M, L_i)=0$.

(b) (slightly different methods) Also true if $[\omega] = a \cdot c_1(M)$
$a>0$.