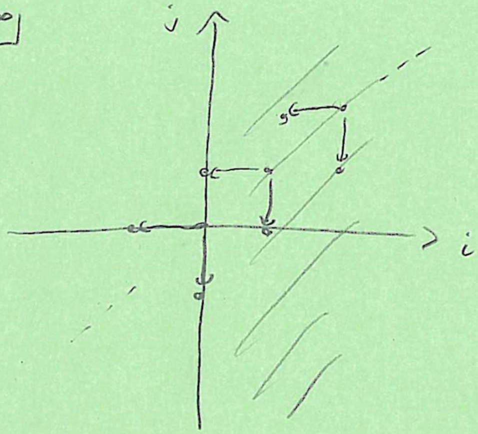
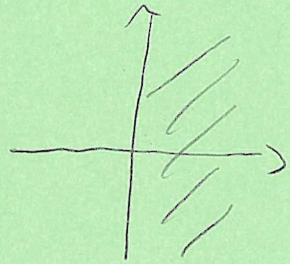
Lecture 11

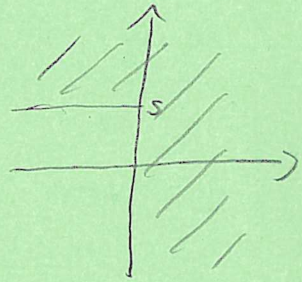
Recall

CFK^∞





$B^+ \simeq CF^+(S^3)$



A_s^+

Maps $v_s^+ : A_s^+ \longrightarrow B^+$ projection onto $C\{i \geq 0\}$

$h_s^+ : A_s^+ \longrightarrow B^+$

- projection onto $C\{j \geq s\}$
- multiplication by U^s
- $C\{j \geq 0\} \simeq C\{i \geq 0\}$

Last time For RHT

On the level of homology

- $v_s^+ = 1 \quad s \geq 1$ $\qquad h_s^+ = U^s \quad s \geq 1$
- $v_0^+ = U$ $\qquad h_0^+ = U$
- $v_{-s}^+ = U^s \quad s \geq 1$ $\qquad h_{-s}^+ = 1 \quad s \geq 1$

$$CF^+(S^3_n(K)) \simeq \mathbb{X}(n)^+ \simeq \text{Cone}\left(\mathbb{A}_s^+ \xrightarrow{\mathbb{D}} \mathbb{B}^+\right)$$

where $\mathbb{A}_s^+ = \bigoplus_{s \in \mathbb{Z}} A_s^+$ $\qquad \mathbb{D}(\{a_s\}) = \{b_s\}$

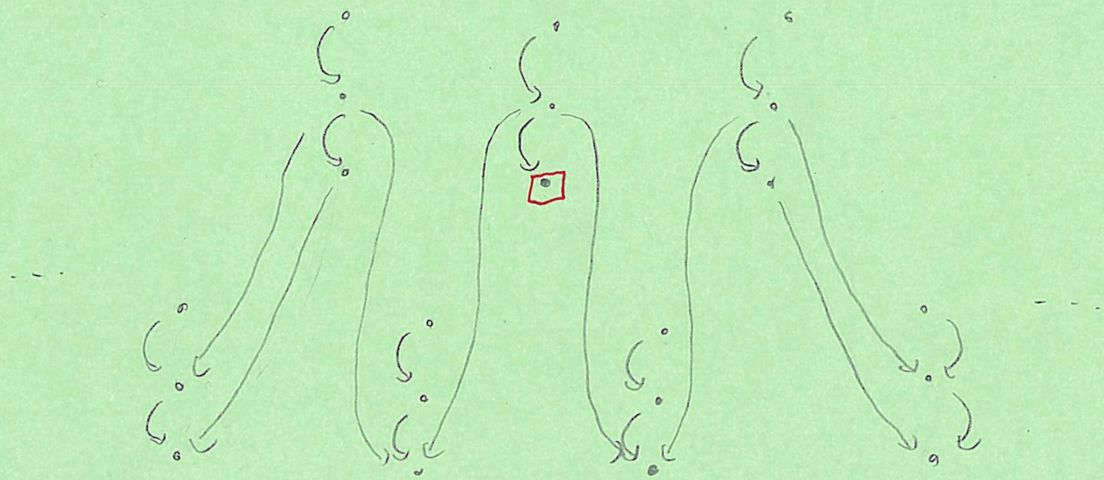
$\mathbb{B}^+ = \bigoplus_{s\in\mathbb{Z}} B^+$ $\qquad$ w/ $b_s = v_s^+(a_s) + h_{s-n}^+(a_{s-n})$

This identification is homogeneous of degree $-d(n,i)$ where

$$d(n,i) = d(L(n,1), [i]) = \max_{\{s \in \mathbb{Z} : s \equiv i \bmod n\}} \frac{1}{4}\left(1 - \left(\frac{n+2s}{n}\right)^2\right)$$

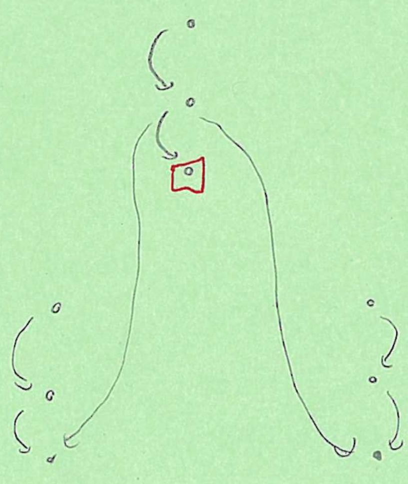
Example $HF^+(S^3_{-1}(RHT))$

$$\begin{tikzcd}[column sep=small]
 & {} \arrow[dl, "1"'] \arrow[dr, "u^2"] & & A_{-1} \arrow[dl, "1"'] \arrow[dr, "u"] & & A_0 \arrow[dl, "u"'] \arrow[dr, "u"] & & A_1 \arrow[dl, "u"'] \arrow[dr, "1"] & & {} \arrow[dl, "u^2"'] \arrow[dr, "1"] & \\
{} & & B_{-2} & & B_{-1} & & B_0 & & B_1 & & {}
\end{tikzcd}$$

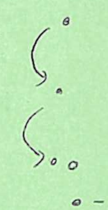


$$\simeq \quad \begin{tikzcd} & A_0 \arrow[dl, "u"'] \arrow[dr, "u"] & \\ B_{-1} & & B_0 \end{tikzcd}$$

$\simeq$



$HF^+(S^3_{+1}(RHT)) \simeq$



Remarks The numbers V_s : $v_s^+ : U\hat{A}_s^+ \longrightarrow U\hat{B}^+$ is U^{V_s} on homology

H_s : $h_s^+ : U\hat{A}_s^+ \longrightarrow U\hat{B}^+$ is U^{H_s} on homology

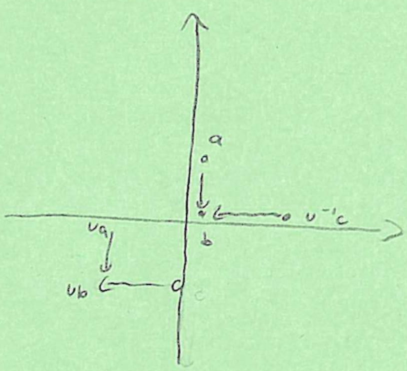
are in fact concordance invts.

Rasmussen $V_0(K) \leq g_4(K)$ (a slice genus bound)

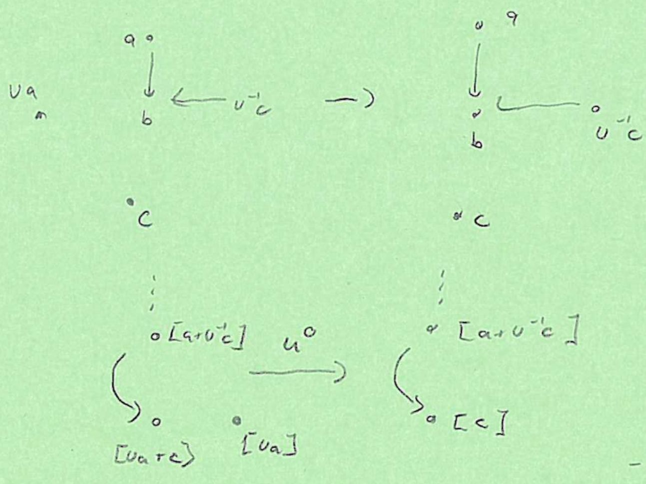
Exercise What is $V_0(Y_t)$? What is $V_0(RHT \# RHT)$?

Note Not a homomorphism

LHT)



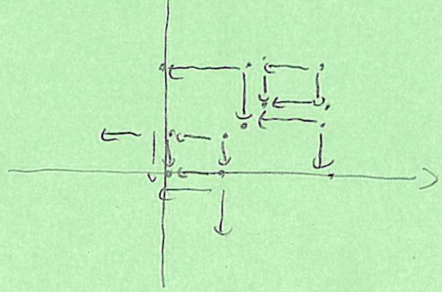
$A_0^+ \longrightarrow B^+$



$\boxed{V_0 = 0}$

Note $CFK^\infty(K_1 \# K_2) = CFK^\infty(K_1) \otimes CFK^\infty(K_2)$ as a tensor product of bifiltered complexes.

Claim $CFK^\infty(RHT \# RHT)$ is

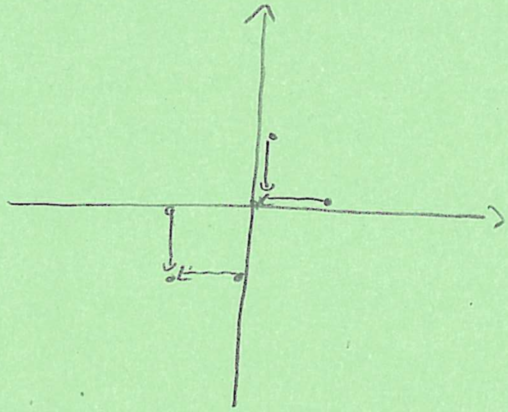


Properties

- $V_K \geq V_{K+1}$
- $H_K \leq H_{K+1}$
- $V_K = 0$ when $K \geq g$
- $V_K \longrightarrow \infty$ as $K \to -\infty$

<u>Note</u> we now have two (claimed) concordance invts from CFK^∞ that lower bound the slice genus

They can be different:



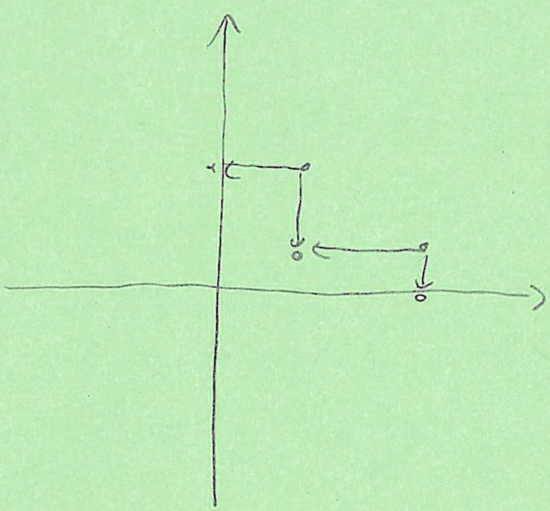
$\tau = -1$

$\nu_0 = 0$

- $\tau : \mathcal{C} \to \mathbb{Z}$ is a (surjective) homomorphism; ν_0 is not.

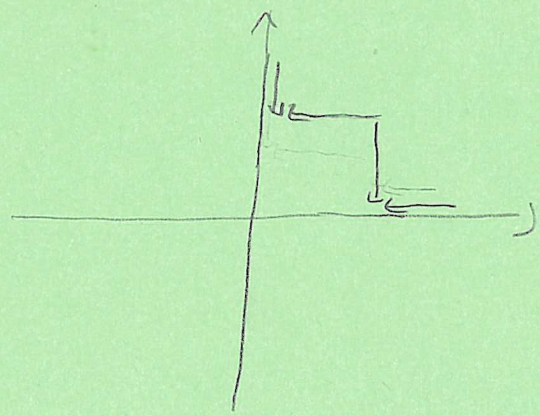
<u>Examples for playing with</u>

- If K is an L-space knot, the coefficients of $\Delta_K(t)$ are all ± 1 and its CFK^∞ is generated by a staircase



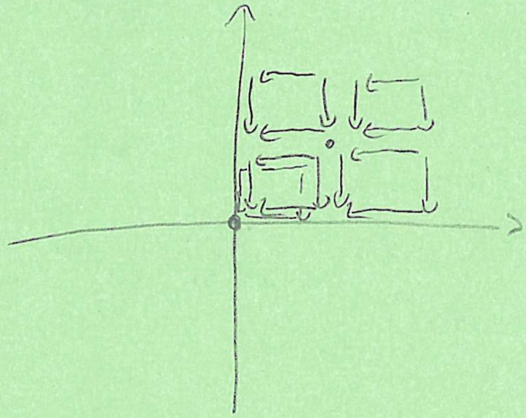
$t^3 - t^2 + 1 + t^{-2} + t^{-3}$

If K is the mirror of an L-space knot we switch the staircase.



For thin knots ($M([x]) - A([x])$ = constant, includes all alternating knots) we get a staircase of step length one plus 1x1 boxes.

eg) 9_{44}



$$\underbrace{\Delta_K(t) = t^{-2} - 4t^{-1} + 7 - 4t + t^2}$$

One can determine the cpx from $\Delta_K(t)$ and $\tau = \frac{-\sigma}{2}$.

Why would such a thing be true?

We consider the cobordism $W_m(K): S^3 \to S^3_m(K)$ and its opposite $-W_m(K): -S^3 \to -S^3_m(K)$. In fact we turn this cobordism around: $W_m'(K): S^3_m(K) \to S^3$.

- Pick a Seifert surface F for K, and let $S \in W_m'(K)$ be the capped off surface.

- Let u be a spin^c-structure on $S^3_m(K)$ extending to s over $W_m'(K)$ st

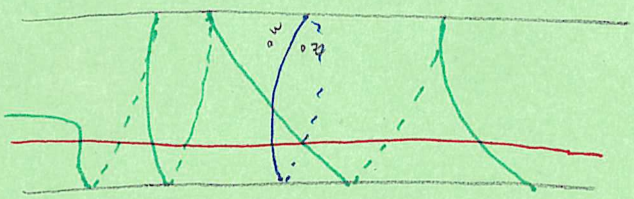
$$\langle c_1(s), [S] \rangle - m \equiv 2t \pmod{2m}$$

Exercise

- This gives an isomorphism $Spin^c(S^3_m(K)) \longrightarrow \mathbb{Z}/m\mathbb{Z}$

$$u \longmapsto t$$

Pick a Heegaard diagram for $W_m'(K)$



$Y_{\alpha\gamma} = S^3_m(K)$

$Y_{\alpha\beta} = S^3$

$Y_{\beta\gamma} = \#_{(g-1)} (S^1 \times S^2)$

$\exists$ a map

$$\Psi^+_{m,s} : CF^+(S^3_m(K), [s]) \longrightarrow C\{\max(i, j-s) \geq 0\}$$

via

$$\Psi^+_{m,s}[\vec{x}, i] = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\{\psi \in \pi_2(x,\theta,y) : n_w(\psi) - n_z(\psi) = s\} \\ \mu(\psi)=0}} \# \mathcal{M}(\psi) [\vec{y}, i - n_w(\psi), i - n_z(\psi)]$$

If n_w is a negative number, then $i - n_z(\psi) \geq s$

Theorem $\exists$ $N >> 0$ st the map $\Psi^+_{m,s}$ induces an isomorphism

$$CF^+(S^3_m(K), t) \simeq A^+_s \quad \text{for } |s| \leq \frac{m}{2},\ s \equiv t \bmod (m),\ m > N$$

In fact $N = 2g - 1$.

Idea of the proof This is the $N >> 0$ large enough that there must be an intersection point in the winding region in this spin^c-structure. Then $\Psi^+_{m,s} = \Psi_0 +$ (lower order terms) and we can conclude Ψ is an isomorphism.

Moreover The cobordism maps associated to the spin^c structures w/ $\langle c_1(\mathfrak{r}_s), [S]\rangle + m = 2s$ and $\langle c_1(\eta_s), [S]\rangle - m = 2s$ are

$$\begin{tikzcd} CF^+(S^3_m(K), t) \arrow[r] \arrow[d, "\Psi^+_{m,s}"'] & CF^+(Y) \arrow[d, "="] \\ A_s^+ \arrow[r, "v_s^+"] & B^+ \end{tikzcd} \qquad \begin{tikzcd} CF^+(S^3_m(K), t) \arrow[r] \arrow[d, "\Psi^+_{m,s}"'] & CF^+(S^3) \arrow[d, "="] \\ A_s^+ \arrow[r, "h_s^+"] & B^+ \end{tikzcd}$$

via post-composing w/ the projection.

This is counting wrt the w basepoint instead of the z basepoint, the spin^c structure changes by PD$[\hat{F}]$.

<u>Thm</u> There is a general surgery exact sequence.

$$\cdots \longrightarrow HF^+(Y_n(K)) \longrightarrow HF^+(Y_{m\cdot n}(K)) \longrightarrow \bigoplus_m HF^+(Y) \longrightarrow \cdots$$

$\exists$ maps $f_2^+ : CF^+(Y_{m\cdot n}(K)) \longrightarrow \bigoplus_m CF^+(Y)$

$f_1^+ : CF^+(Y_n(K)) \longrightarrow CF^+(Y_{m\cdot n}(K))$

We use this for m and $m(k+1)$ for large k.

$$f_2^+ : \underbrace{CF^+(S^3_{m(k+1)})}_{\bigoplus_{s} A_s^+} \longrightarrow \bigoplus_m CF^+(S^3)$$

Remaining term is $CF^+(S^3_m(K))$

The proof goes by checking that the mapping cone of f_2^+ can be replaced w/ the mapping cone of $\mathbb{A}^+ \longrightarrow \mathbb{B}^+$: for large k this is a truncation of the complex...

Gradings are such that surgeries work out correctly on the unknot.

<u>Ni-Wu</u> Let $n > 0$. Then $d(S^3_n(K), [0]) = d(L(n,1), [0]) - 2V_0$.