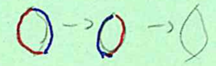
Lecture 10 ⑦

Orientation Reversal

Note $d(Y,t) = d(Y,\bar{t})$

$d(Y,t) = -d(-Y,t)$

Using $(\Sigma, \alpha, \beta, z)$ and $(-\Sigma, \alpha, \beta, z)$ } This reverses the direction of every disk



There is a natural pairing $CF^{\infty}(Y,t) \times CF^{\infty}(-Y,t) \longrightarrow \mathbb{Z}$

$$\langle [x,i] \;,\; [y,j] \rangle \longmapsto \begin{cases} 1 & \text{if } x = y \text{ and } i+j+1 = 0 \\ 0 & \text{otherwise} \end{cases}$$

w/ the properties that $\langle \xi, \partial^{\infty}_{-Y} \eta \rangle = \langle \partial^{\infty}_{Y} \xi, \eta \rangle$

$\langle \xi, U\eta \rangle = \langle U\xi, \eta \rangle$

$\Rightarrow \exists$ a pairing $\langle , \rangle : HF^{\infty}(Y,s) \otimes HF^{\infty}(-Y,s) \longrightarrow \mathbb{Z}$

$\langle , \rangle : HF^{+}(Y,s) \otimes HF^{-}(-Y,s) \longrightarrow \mathbb{Z}$

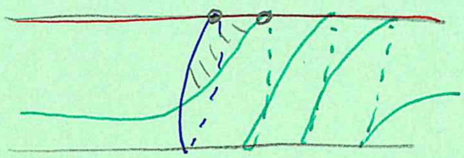
$\Rightarrow \exists$ duality isomorphisms

$$\begin{array}{ccc} HF^{\infty}_{k}(Y,t) & \xrightarrow{\pi_k} & HF^{+}_{k}(Y,t) \\ \wr \downarrow D^{\infty} & & \wr \downarrow D^{+} \\ \mathbb{Z} \cong HF_{\infty}^{-k-2}(-Y,t) & \xrightarrow{\iota^{-k-2}} & HF_{-}^{-k-2}(-Y,t) \end{array}$$

Universal coefficients + HF^{∞} is free $\Rightarrow$ ι^{-k-2} has rank one $\Leftrightarrow$ $\iota_{-k-2} : HF^{-}_{-k-2}(-Y,t) \to HF^{\infty}_{-k-2}(-Y,t)$ is nontrivial. So $d(Y,t) = -d(-Y,t)$.

Surgery Formulas

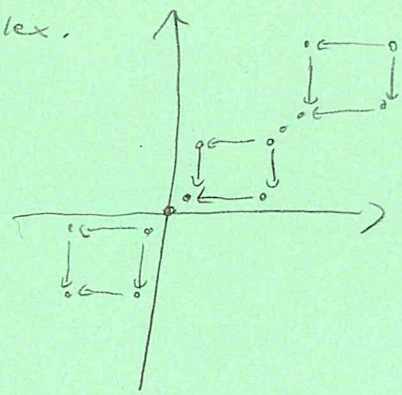
Idea Recall there is a diagram for (S^3, K) w/ β a meridian of the knot, we can replace w/ a suitably-wrapped longitude.



There is a small triangle map $CFK^+(S^3, K) \longrightarrow CF^+(Y, s)$.

We look at subquotient complexes of CFK^∞ and maps between them.

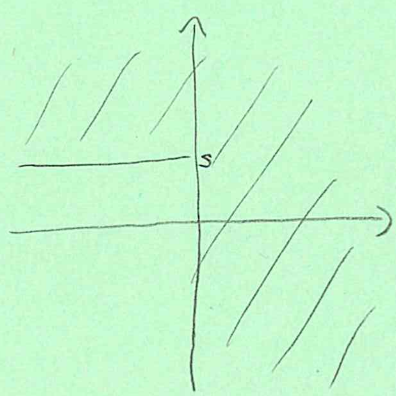
Recall $CFK^\infty(K)$ is a $\mathbb{Z} \oplus \mathbb{Z}$-filtered complex.



Defn $B_s^+ = \{C\ i \geq 0\} = CF^+(S^3)$

In general $C\{i \geq 0\} \simeq C\{j \geq 0\}$.

$A_s^+ = C\{\max(i, j-s) \geq 0\}$



Define maps $v_s^+ : A_s^+ \longrightarrow B^+$: projection onto $C\{i \geq 0\}$.

$h_s^+ : A_s^+ \longrightarrow B^+$: • projection onto $C\{j \geq s\}$

• multiplication by U^s to $C\{j \geq 0\}$

• [illegible]equivalence w/ $C\{i \geq 0\}$

Defn $\mathbb{A}^+ = \bigoplus_{s \in \mathbb{Z}} A_s^+$

$\mathbb{B}^+ = \bigoplus_{s \in \mathbb{Z}} B_s^+$ } All summands are the same

$D_n^+ : \mathbb{A}^+ \longrightarrow \mathbb{B}^+$

$\{a_s\}_{s \in \mathbb{Z}} \longrightarrow \{b_s\}_{s \in \mathbb{Z}}$ $\qquad b_s = h_{s-n}^+(a_{s-n}) + v_s^+(a_s)$

$s - n = 0$

$s = n$

Define $\mathbb{X}^+(n)$ be the mapping cone of D_n^+, i.e. $\mathbb{A}^+ \oplus \mathbb{B}^+$ w/

$\partial(\{a_s\}, \{b_s\}) = (\partial_{\mathbb{A}^+}\{a_s\},\ D_n^+\{a_s\} + \partial_{\mathbb{B}^+}\{b_s\})$ over $\mathbb{F}_2$.

Thm The homology of this mapping cone is $\simeq HF^+(S_n^3(K))$.

Note this derives from

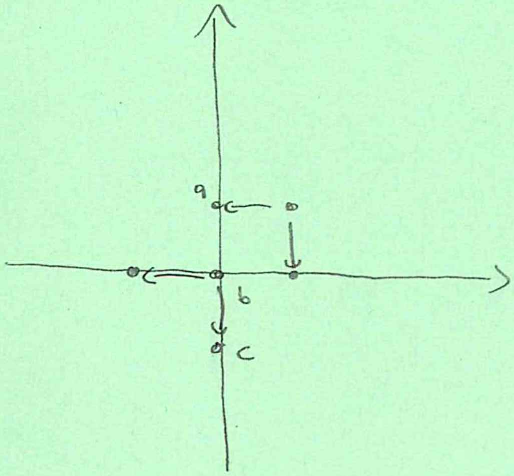
Thm $K \subseteq Y$ a homology sphere. $\exists\, N$ st $\forall\, m \geq N$ and all $t \in \mathbb{Z}_m$,

$CF^+(Y_m(K), t) \simeq A_s^+$ for $s \equiv t \mod n$.

Note In fact $N \geq 2g-1$ suffices in general.

$\mathcal{T}^+ = \mathbb{F}[U, U^{-1}] / U\mathbb{F}[U]$

Example $CFK^\infty(RHT)$

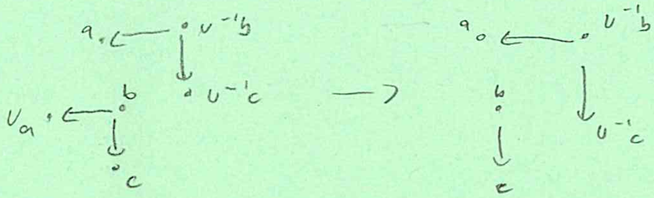


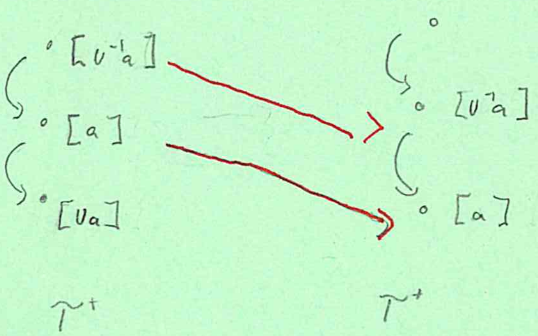
$$B^+ \simeq \begin{tikzcd} a & U^{-1}b \arrow[l] \arrow[d] \\ b \arrow[d] & U^{-1}c \\ c & \end{tikzcd}$$

$H_*(B^+) \simeq \mathcal{T}^+$

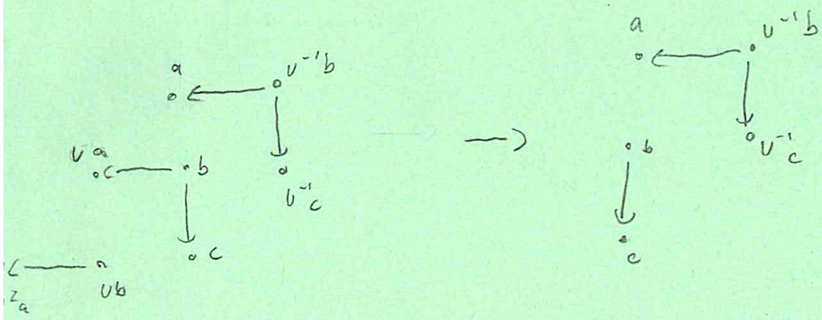
Examples

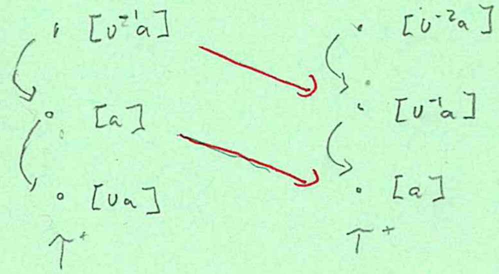
$v_0^+ : A_0^+ \longrightarrow B^+$





$v_{-1}^+ : A_0^+ \longrightarrow B^+$





On the level of homology

- $v_s^+ = 1 \quad s \geq 1 \quad A_s^+ = B^+$
- $v_0^+ = U$
- $v_{-1}^+ = U$
- $v_{-2}^+ = U^2$

$\vdots$

- $v_{-n}^+ = U^n$

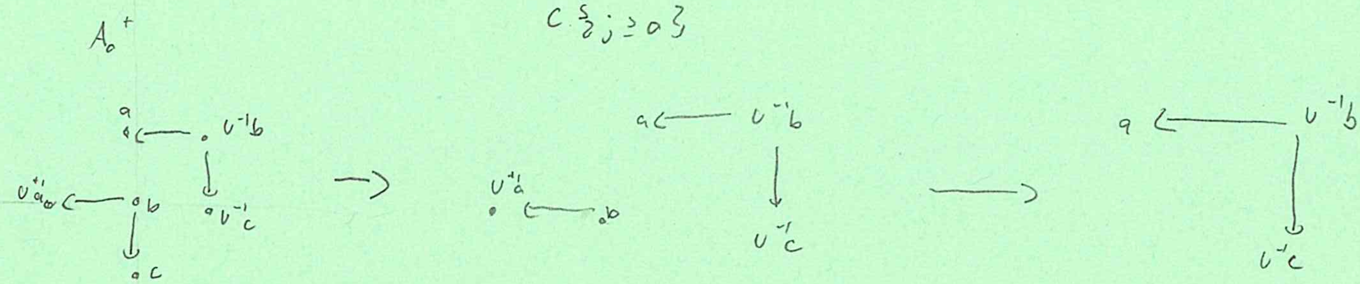
$h_s^+ = U^s \quad s \geq 1$

$h_0^+ = U$

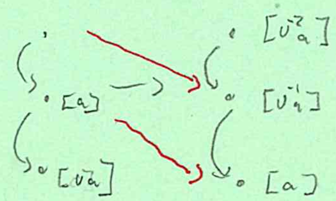
$h_s^+ = 1 \quad s \leq -1$

(9)

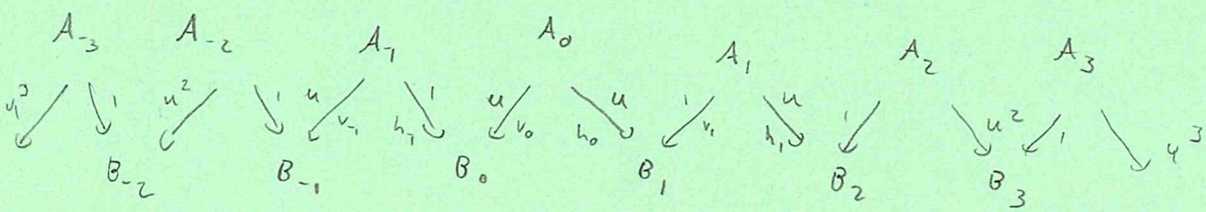
$\underline{h_0^+}$ $A_0^+ \longrightarrow B^+$



$T^+ \longrightarrow T^+$



$\underline{\text{Example}}$ $CF^+(S^3_{+1}(RHT))$



$\underline{\text{Note}}$ $H_*(B_s^+)$ is a divisible $\mathbb{F}[U]$ module, hence injective $\rightsquigarrow$ since $\mathbb{F}[U]$ is a pid, $H_*(B_s^+)$ is injective. Moreover cokernel of $\mathbb{D}$ must then be injective, as it is a quotient of an injective module $H_*(B_s^+)$

So any extension of coker$(\mathbb{D}_*^+)$ by some module ker$(\mathbb{D}_*^+)$ is trivial

$\rightsquigarrow$ we can pass to homology above.

$\underline{\text{Not true for } \mathbb{F}[U] \text{ over } \mathbb{F}[U]}$

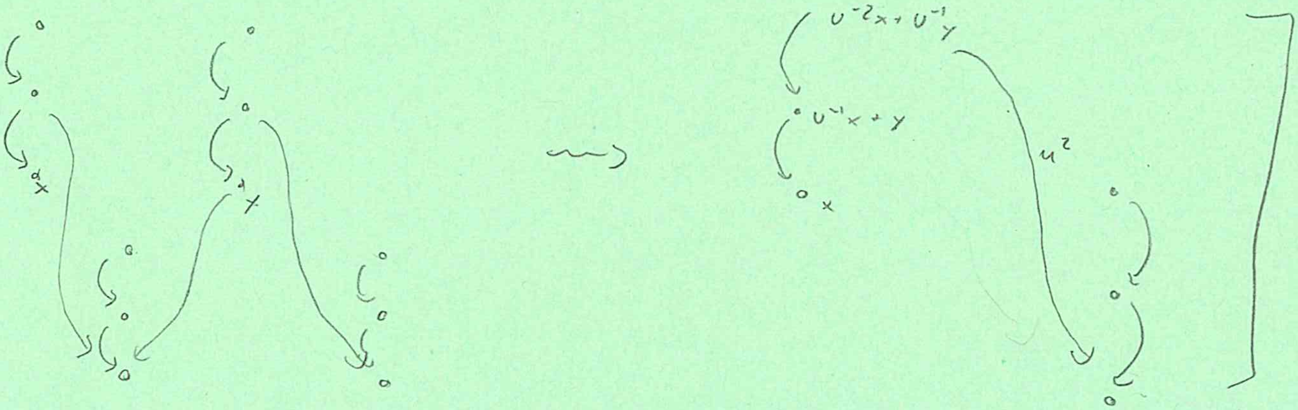
$0 \to \underbrace{\mathbb{F}[U]}_{\text{nope}} \xrightarrow{U} \mathbb{F}[U] \to \mathbb{F} \to 0$

$\to H_*(\mathbb{B}^+) \to H_*(\mathbb{X}^+) \to H_*(\mathbb{A}^+) \to H_*(\mathbb{B}^+) \to \ldots$

$0 \to \text{coker}(\mathbb{D}^+) \to H_*(\mathbb{X}^+) \to \ker(\mathbb{D}^+) \to 0$

Moreover, we can replace

$$\begin{tikzcd}[column sep=small] \overset{x}{A_0} \arrow[dr, "u"'] & & \overset{y}{A_1} \arrow[dl, "1"'] \arrow[dr, "u"] & \\ & B_1 & & B_2 \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} "A_0" \arrow[dr, "u \cdot (1)^{-1} u = u^2"] & \\ & B_2 \end{tikzcd}$$

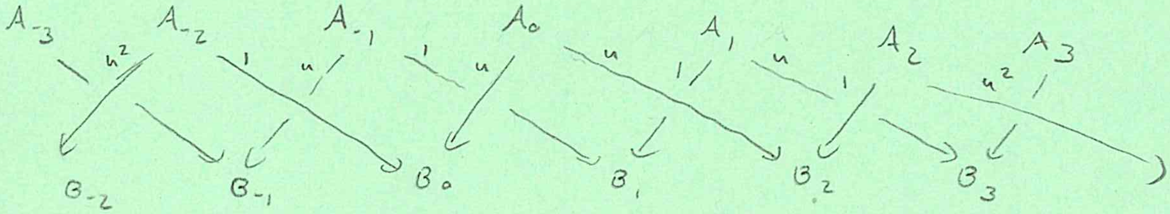


Zig-zag lemma

$$\rightsquigarrow \begin{tikzcd} "A_0" \arrow[dr, "u^4"] & \\ & B_3 \end{tikzcd}$$

we conclude that $HF^+(S^3_{+1}(RHT)) = \mathcal{T}^+$

$HF^+(S^3_2(RHT))$



Break into pieces

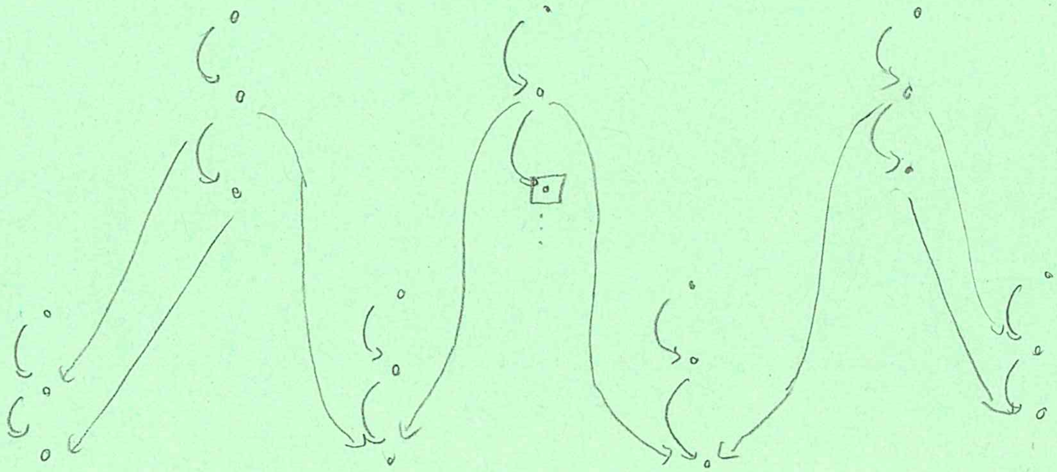
$$\begin{tikzcd}[column sep=small]
 & \overset{x_{-3}}{A_{-3}} \arrow[dl] \arrow[dr] & & \overset{x_{-1}}{A_{-1}} \arrow[dl, "u"'] \arrow[dr, "1"] & & \overset{x_1}{A_1} \arrow[dl, "1"'] \arrow[dr, "u"] & & \overset{x_3}{A_3} \arrow[dl, "1"'] \arrow[dr] & \\
\cdots & & B_{-1} & & B_1 & & B_3 & & \cdots
\end{tikzcd} \quad \rightsquigarrow \tau^+$$

τ^+ generated by $U_x^{-k}(\cdots + U^3 + U + 1 + 1 + U + U^3 + \cdots)$

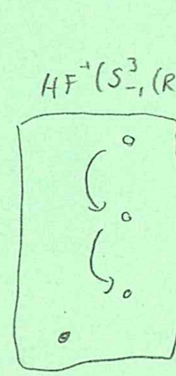
$$\begin{tikzcd}[column sep=small]
 & A_{-2} \arrow[dl, "u^2"'] \arrow[dr, "1"] & & A_0 \arrow[dl, "u"'] \arrow[dr, "u"] & & A_2 \arrow[dl, "1"'] \arrow[dr, "u^2"] & \\
\cdots & & B_0 & & B_2 & & \cdots
\end{tikzcd} \quad \simeq H^*(A_0^+) \simeq \tau^+$$

$HF^+(S^3_{-1}(RHT))$

$$\begin{tikzcd}[column sep=small]
\cdot \arrow[dr, "u^2"] & & A_{-1} \arrow[dl, "1"'] \arrow[dr, "u"] & & A_0 \arrow[dl, "u"'] \arrow[dr, "u"] & & A_1 \arrow[dl, "u"'] \arrow[dr, "1"] & & \cdots \arrow[dl, "u^2"'] \\
 & B_{-2} & & B_{-1} & & B_0 & & B_1 &
\end{tikzcd}$$



$HF^+(S^3_{-1}(RHT))$



The numbers V_s: $v_s : A_s^+ \longrightarrow B^+$ is U^{V_s} on homology

H_s: $h_s : A_s^+ \longrightarrow B^+$ is U^{H_s} on homology

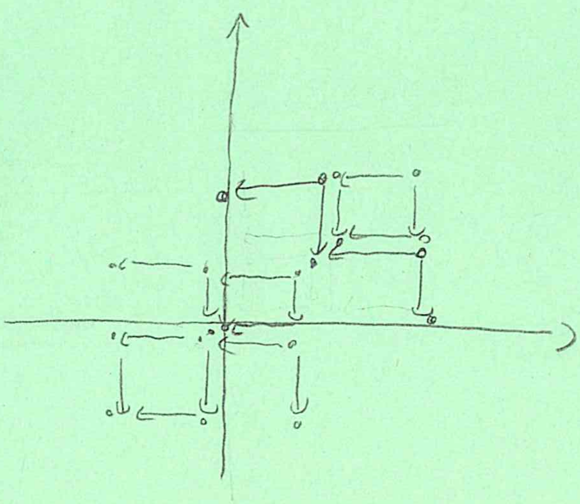
are in fact concordance invts.

• Rasmussen $V_0(K) \le g_4(K)$ (a slice genus bound)

Exercise? What is $V_0(RHT \# RHT)$?

$CFK^\infty(K_1 \# K_2) = CFK^\infty(K_1) \otimes CFK^\infty(K_2)$ as a tensor product of bifiltered complexes

Claim $CFK^\infty(RHT \# RHT)$ is



Properties

• $V_k \ge V_{k+1}$ • $V_k = 0$ when $k \ge g$

$H_k \le H_{k+1}$ • $V_k \to \infty$ as $k \to \infty$

• $V_k = H_{-k}$