Lecture 1

Policies & Practicalities

- Syllabus
- There will be talks! Come discuss in March.

What is Heegaard Floer homology?

Structural Overview

Y - three-manifold (often restrict to $\mathbb{Q}HS^3$ or $\mathbb{Z}HS^3$

$\mathfrak{s}$ - spin^c structure] will discuss shortly what this is,
but $\text{Spin}^c(Y) \cong H^2(Y)$

$(Y,\mathfrak{s}) \rightsquigarrow CF^\circ(Y,\mathfrak{s}) \rightsquigarrow HF^\circ(Y,\mathfrak{s}) \rightsquigarrow$ Various numerical invariants

- $\circ \in \{+, -, \infty, \wedge, \text{red}\}$
- Chain cpx over $\mathbb{F}_2[U]$
- Relatively graded in general, absolutely graded if Y is torsion. (so eg if Y is a $\mathbb{Q}HS^3$)

- Module over $\mathbb{F}[U]$
- $HF^\circ(Y) = \bigoplus_{\mathfrak{s} \in \text{Spin}^c(Y)} HF^\circ(Y,\mathfrak{s})$

These complexes have short exact sequences:

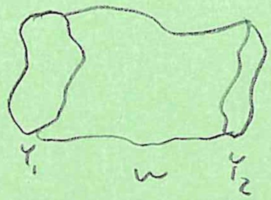
$$0 \to CF^-(Y) \xrightarrow{\cdot U} CF^\infty(Y) \to CF^+(Y) \to 0$$

$$0 \to CF^-(Y) \xrightarrow{\cdot U} CF^-(Y) \to \widehat{CF}(Y) \to 0$$

$$0 \to \widehat{CF}(Y) \to CF^+(Y) \xrightarrow{\cdot U} CF^+(Y) \to 0$$

w/ corresponding long exact sequences in homology.

Cobordisms induce maps



- $F_W : CF^\circ(Y_1) \longrightarrow CF^\circ(Y_2)$

  w/ predictable properties

- Composition of cobordisms corresponds to composition of maps

- Recovers 4D information similar to the SW invt.

Knots (and Links)

Y - 3-mfd, usually a $\mathbb{Z}HS^3$

K - nulhomologous knot

- K induces a Filtration on $CF^\circ(Y)$
- Homology of the associated graded is $HFK^\circ(Y, K)$
- Eg there is a spectral sequence $HFK^\circ(Y,K) \Rightarrow HF^\circ(Y)$
- In the special case $Y = S^3$, $\widehat{HFK}$ detects the genus & whether the knot is Fibred, and recovers $\Delta_K(t)$.
- $CFK^\infty \rightsquigarrow$ various invariants of knot concordance

Surgery Formulas

- $CFK^\infty(K)$ determines $CF^\circ(S^3_{p/q}(K))$
- Key to using Heegaard Floer for hyperbolic examples.

Historical Context

Physics - The classical equations of motion often come w/ symmetries

- Translations of $\mathbb{R}^3 \rightsquigarrow$ momentum
- Rotations of $\mathbb{R}^3$ $(O(3)) \rightsquigarrow$ angular momentum

- Maxwell's equations for electromagnetism have a $U(1)$ gauge symmetry

  - Mathematically, corresponds to looking at connections of a principal $U(1)$ bundle over $\mathbb{R}^{3,1}$.

- Yang-Mills eqns Elliptic pdes, generalizing Maxwell to higher-rank bundles

  - $SU(2) \rightsquigarrow$ weak interactions in particle physics
  - $SU(3) \rightsquigarrow$ strong interactions
  - $SU(3) \times SU(2) \times U(1) \rightsquigarrow$ std particle physics model
    "everything but gravity"

Mathematical Gauge Theory Usually for $SU(2)$ or $SO(3) = PSU(2)$

- Typically studies "anti-self-dual" Yang-Mills equations on principal $SU(2)$ bundles. Solutions are instantons

  - Various difficult progress in the '70s counting instantons
    - S^4: 1978, Atiyah-Drinfield-Hitchin-Manin
    - Riemann surfaces: 1978 Atiyah-Bott

• 1982 Donaldson : YM equations on arbitrary M^4 ⇝ topological applications

• Diagonalizability Thm M^4 closed, oriented, simply cnd smooth w/ intersection form Q. If Q is definite, then Q is diagonalizable.

over $\mathbb{Z}$. over $\mathbb{R}$ is always true.

Freedman Any Q w/ $\det Q = 1$ can be realized as the intersection form of a topological mfd.

⇝ Many examples of nonsmoothable topological mfds, eg E_8

Donaldson Invariant Count instantons ⇝ examples of 4-mfds which are homeomorphic but not diffeomorphic.

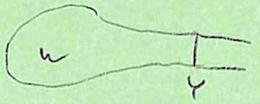
Dimensional Reduction Equations for connections / forms on $M^n = X^{n-1} \times \mathbb{R}$
⇝ impose $\mathbb{R}$-invariance ⇝ equations on X^{n-1}

① ASD Y-M on $Y^3 \times \mathbb{R}$ ⇝ Floer homology '88

$Y^3 \longrightarrow CF_*(Y)$ freely generated by $\mathbb{R}$-invt solutions on $Y \times \mathbb{R}$

⇝ homology $I^\#(Y)$ instanton Floer homology

• Wanted for defining Donaldson invts on mfds w/ bdy



for some cut and paste computation

Simultaneously, Floer: Many other variants of Floer homology, including <u>Lagrangian Floer homology</u>

$(M, L_0, L_1) \rightsquigarrow CF(L_0, L_1) \rightsquigarrow HF(L_0, L_1)$. Modelled on morse theory, counts pseudohol'c curves. Inspired by Gromov.

(II) Other dim'l reductions

$\boxed{4D} \rightsquigarrow \boxed{3D} \rightsquigarrow \boxed{2D} \rightsquigarrow \boxed{1D}$

ASD Y-M, Bogomolny, Hitchin, Nahm

<u>More gauge-invariant eqns</u>

(1) Seiberg-Witten equations ('94) on M^4

- U(1) instead of SU(2); more computable replacement for Yang-Mills
- Many topological applications

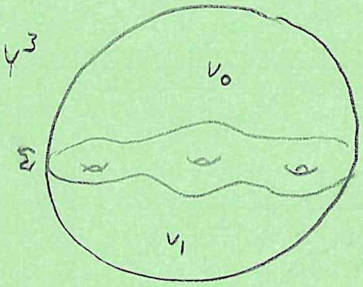
(2) Dim'l reductions

- 3D Seiberg-Witten eqns
  $\rightsquigarrow$ Kronheimer-Mrowka HM monopole Floer homology replaces $I^\#(Y)$
- 2D Vortex eqns $\rightsquigarrow$ Heegaard Floer homology

(3) Vafa-Witten, Kapustin, etc

Specifically

Atiyah-Floer conjecture



(Atiyah-Bott)
Has a natural
Symplectic
Form

$I_*(Y)$ = Lagn Floer homology $HF(L_0, L_1) \in M_{Flat}(\Sigma)$

$HM(Y)$ = Lagn Floer homology $HF(L_0, L_1)$ in $Sym^k(\Sigma)$

Ozsváth and Szabó made a guess about what goes here based on what they could compute

$\rightsquigarrow$ Heegaard Floer turned out to be ridiculously computable by the standards of this whole story.

What does it mean to split a 3-mfd like this? Well, that returns us to practical-land....

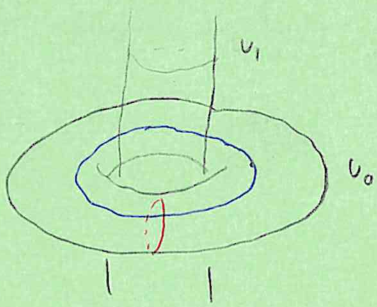
· Y a closed oriented 3-mfd.

<u>Defn</u> A genus g handlebody U is diffeomorphic to a regular nbhd of a bouquet of g circles in $\mathbb{R}^3$ (i.e. it is a g-holed torus). The boundary of U is an oriented surface w/ genus g.

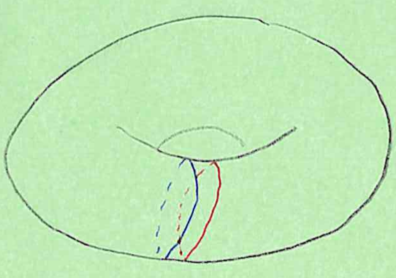
A <u>Heegaard decomposition</u> of Y consists of two genus g handlebodies U_0, U_1 and a homeomorphism $\varphi : \partial U_0 \to \partial U_1$ such that $Y = U_0 \cup_\varphi U_1$.

<u>Examples</u> $S^3 = (S^1 \times D^2) \coprod_\varphi (S^1 \times D^2)$ where $\varphi : S^1 \times S^1 \to S^1 \times S^1$

$$(\phi, \theta) \mapsto (\theta, \phi)$$



· $S^1 \times S^2 = (S^1 \times D^2) \coprod_\varphi (S^1 \times D^2)$ where $Id = \varphi : S^1 \times S^1 \to S^1 \times S^1$



• $L(p,q)$

$S^3 = \{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}$

If $(p,q)=1$, $1 \le q < p$. Then $L(p,q)$ is given by modding out by

$F: (z,w) \longmapsto (\alpha z, \alpha^q w)$ w/ $\alpha = e^{2\pi i/p}$

$U_0 = |z| \le \frac{1}{2}$ $\quad U_1 = |z| \ge \frac{1}{2}$ are preserved and their quotients are tori.

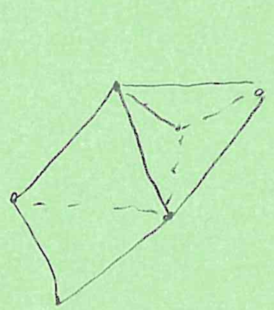
$\rightsquigarrow$ Heegaard decomposition

<u>Exercise</u> So far these are all genus one. Can anyone give me an example of a mfd we couldn't get a genus one surface for?

<u>Thm</u> (~'98~) Let Y be an oriented closed 3-dimensional mfd. Then Y admits a Heegaard decomposition.

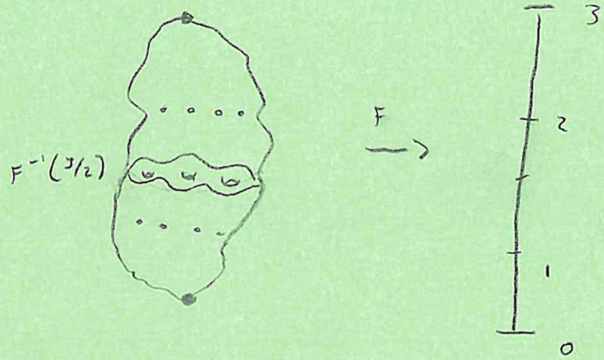
<u>Proofs</u>

① We can triangulate any 3-mfd. Then a small nbhd of the edges and vertices, which form a graph, is a g-handlebody. The complement is a nbhd of a graph on the centers of the triangles and tetrahedrons.



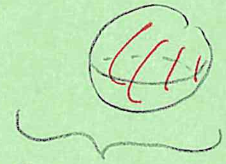
Alternative point of view:

Given Y, we have a self-indexing Morse function



For purposes of this discussion, say it has a single critical pt at each of index 0 and 3.

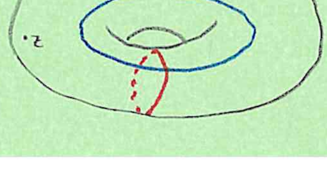
$\rightsquigarrow$ + 3-handles



This attaching map is unique up to isotopy

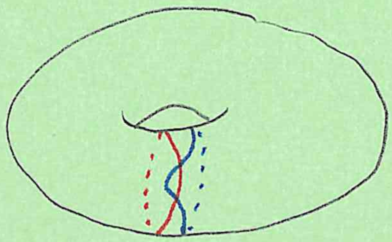
Upshot: To describe a Heegaard decomposition, one thinks of each handlebody as the top half of the decomposition and specifies where to put the 2-handles by adding circles.

<u>Example</u> S^3



Note that this is equivalently the intersection of the ascending mfds of index one critical pts and the intersection of the descending mfds of index two

$S^1 \times S^2$



$L(3,1)$