MTH 961: Suggested Exercises for Week 7

1. Verify that the derived couple of an exact couple is still an exact couple.

2. Verify that the differential on the $E^2$ page of the spectral sequence arising from a double complex is well-defined.

3. Suppose you have a first-quadrant spectral sequence over $\mathbb{Z}$ of cohomological type which is known to converge to a module $H^*$, and whose $E_2$ page is

$$E_2^{p,q} = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } (p, q) = (0, 0), (0, 4), (2, 3), (3, 2), \text{ or } (6, 0) \\ 0 & \text{otherwise} \end{cases}$$

Determine all possible candidates for $H^*$.

4. Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration with $B$ simply-connected and $F$ having the homology of $S^n$. Show that there is a Gysin exact sequence

$$\cdots \rightarrow H_p(E) \rightarrow H_p(B) \rightarrow H_{p-n-1}(B) \rightarrow H_{p-1}(E) \rightarrow \cdots$$

(Hint: Two different places on the $E^\infty$ page of the spectral sequence contribute to $H_p(E)$; combine them into one.)

5. Use the Serre spectral sequence to give an (overpowered) reproof of the Leray-Hirsch theorem:

**Theorem 0.1.** Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration. Suppose there are classes $c_j$ in $H^*(E; \mathbb{Q})$ whose restrictions $i^*(c_j)$ form a basis for $H^*(F; \mathbb{Q})$. Then $H^*(E; \mathbb{Q}) \cong H^*(F; \mathbb{Q}) \otimes H^*(B; \mathbb{Q})$ as a vector space.

(This problem uses the cup product structure on the cohomology Serre spectral sequence, which we will talk about on Thursday. For the usual proof, see Hatcher’s Algebraic Topology Section 4D.)