MTH 961: Suggested Exercises for Week 7

1. Verify that the derived couple of an exact couple is still an exact couple.

2. Verify that the differential on the E^2 page of the spectral sequence arising from a double complex is well-defined.

3. Suppose you have a first-quadrant spectral sequence over $\mathbb{Z}$ of cohomological type which is known to converge to a module H^*, and whose E_2 page is

$$E_2^{p,q} = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } (p,q) = (0,0), (0,4), (2,3), (3,2), \text{ or } (6,0) \\ 0 & \text{otherwise} \end{cases}$$

   Determine all possible candidates for H^*.

4. Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration with B simply-connected and F having the homology of S^n. Show that there is a *Gysin exact sequence*

$$\cdots H_p(E) \to H_p(B) \to H_{p-n-1}(B) \to H_{p-1}(E) \to \cdots$$

   (Hint: Two different places on the E^∞ page of the spectral sequence contribute to $H_p(E)$; combine them into one.)

5. Use the Serre spectral sequence to give an (overpowered) reproof of the Leray-Hirsch theorem:

   Theorem 0.1. *Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration. Suppose there are classes c_j in $H^*(E;\mathbb{Q})$ whose restrictions $i^*(c_j)$ form a basis for $H^*(F;\mathbb{Q})$. Then $H^*(E;\mathbb{Q}) \simeq H^*(F;\mathbb{Q}) \otimes H^*(B;\mathbb{Q})$ as a vector space.*

   (This problem uses the cup product structure on the cohomology Serre spectral sequence, which we will talk about on Thursday. For the usual proof, see Hatcher's Algebraic Topology Section 4D.)