MTH 961: Suggested Exercises for Week 15

1. Prove that the coefficient homomorphism $H^i(B;\mathbb{Z}) \to H^i(B;\mathbb{Z}_2)$ maps the total Chern class of a complex bundle ω to the total Stiefel-Whitney class of the underlying real vector bundle $\omega_{\mathbb{R}}$. In particular, show that the odd Stiefel-Whitney classes of $\omega_{\mathbb{R}}$ are zero.

2. The Stiefel manifold $V_{n-q}(\mathbb{C}^n)$ is $2q$-connected and has $\pi_{q-1}(V_{n-q}(\mathbb{C}^n)) \simeq \mathbb{Z}$. Using this, show that $c_{q-1}(\omega)$ is the obstruction to extending $n-q$ nowhere zero sections of a complex bundle ω over the $2q+2$th skeleton of the base.

3. Prove the *splitting principle*: Given an n-plane bundle ξ (real or complex) over a paracompact base B, there exists a space Y and a map $f\colon Y \to X$ such that $f^*\colon H^*(X) \hookrightarrow H^*(Y)$ is injective and $f^*(\xi) = L_1 \oplus L_1 \oplus \cdots \oplus L_n$, where the L_i are line bundles. (Hint: Iterate the strategy we used in class to split off a single line bundle.)

4. Given a complex vector bundle ω, let $\Lambda^n\omega$ be the vector bundle such that, if F_b is the fibre of ω over b, the fibre of $\Lambda^n\omega$ over b is the space of alternating multilinear maps $F_b^* \times \cdots \times F_b^* \to \mathbb{C}$. Use the splitting principle to show that $c_1(\omega) = c_1(\Lambda^n\omega)$.

5. Let $S_d = \{[z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 : \sum z_i^d = 0\}$ be a degree d hypersurface in $\mathbb{CP}^3$. Compute the Chern classes of the tangent bundle and normal bundle of S_d. (Hint: Start with the normal bundle.)