MTH 961: Suggested Exercises for Week 13

1. Let \( \gamma^n \) be the canonical \( n \)-plane bundle over \( \text{Gr}_n(\mathbb{R}^\infty) \). Prove that \( \gamma^n \oplus \gamma^n \) is orientable and has \( w_{2n}(\gamma^n \oplus \gamma^n) \neq 0 \), hence \( e(\gamma^n \oplus \gamma^n) \neq 0 \). If \( n \) is odd, show that \( 2e(\gamma^n \oplus \gamma^n) = 0 \).

2. Consider the complex Grassmannian \( \text{Gr}_n(\mathbb{C}^\infty) \). This has a canonical oriented \( 2n \)-plane real bundle \( \xi^{2n} \) (by regarding each complex \( n \)-dimensional subspace as an oriented real \( n \)-dimensional subspace). Show the restriction of this bundle to the real subspace \( \text{Gr}_n(\mathbb{R}^\infty) \) is isomorphic to \( \gamma^n \oplus \gamma^n \), and conclude that \( e(\xi^{2n}) \neq 0 \).

3. Let \( S^n \) be the unit sphere and \( A \subset S^n \times S^n \) be the anti-diagonal, consisting of pairs of antipodal unit vectors. Using stereographic projection, show that \( TS^n \) is canonically homeomorphic to \( S^n \times S^n - A \). Hence show that \( H^\ast(E, E_0) \simeq H^\ast(S^n \times S^n, S^n \times S^n - \Delta) \simeq H^\ast(S^n \times S^n, A) \subset H^\ast(S^n \times S^n) \). Now, if \( n \) is even, show that \( e(TS^n) \) is twice a generator of \( H^n(S^n; \mathbb{Z}) \). Conclude that \( TS^n \) possesses no nontrivial subbundles.

[This is worked out in Hatcher’s Vector Bundles book; you are encouraged to go read through it if you get stuck.]

4. Use last week’s exercises to give an example of a bundle with vanishing Euler class but no nowhere zero section.

5. A construction of the Steenrod squares. Everything below is in \( \mathbb{Z}_2 \)-coefficients. Notation mostly chosen to match Hatcher’s Algebraic Topology Section 4L, although he does all the other Steenrod prime powers at the same time as the squares.

In theory, a squaring operation would involve \( X \times X \), but it’s actually easier to work with \( X \wedge X \), the smash product. This has a \( \mathbb{Z}_2 \) action generated by the map \( T \) (for transposition) that interchanges the factors; the basepoint \( x_0 \) in the smash product is a fixed point of the action. Consider the Borel construction

\[
\Gamma X = (X \wedge X) \times_{\mathbb{Z}_2} S^\infty := \left( (X \wedge X) \times S^\infty \right)/\left( (x_1, x_2), z \sim (x_2, x_1, -z) \right).
\]

There is a fibre bundle \( (X \wedge X) \to \Gamma X \to \mathbb{RP}^\infty \). Furthermore, since \( x_0 \in X \times X \) is a fixed point, we have a basepoint section \( \mathbb{RP}^\infty \to Y \). We let the quotient of \( \Gamma X \) by this copy of \( \mathbb{RP}^\infty \) be \( \Delta X \), which is now a basepointed space. If we restrict this entire construction to \( S^1 \), we get subspaces \( \Gamma^1 X \) and \( \Lambda^1 X \). (Exercise: All of these constructions are natural, and if \( X \) has the structure of a CW complex, so do \( \Gamma X \), \( \Delta X \), \( \Gamma^1 X \), and \( \Lambda^1 X \).)

Now, there is an isomorphism

\[
H^\ast(\Delta^1 X) \to H^\ast(\Gamma^1 X) \otimes H^\ast(X).
\]

(Exercise: Convince yourself this is true, if necessary.) In particular, we can think about \( \alpha \otimes \alpha \) as an element of \( H^{2n}(\Delta X) \). Our goal is to construct an element \( \lambda(\alpha) \in H^{2n}(\Delta X) \) that restricts to \( \alpha \otimes \alpha \) on each fibre \( X \times X \subset \Delta X \). By naturality, it suffices to construct a suitable \( \lambda(\iota) \in H^{2n}(K(\mathbb{Z}_2, n)) \), where \( \iota \) is the fundamental class in \( H^n(K(\mathbb{Z}_2, n)) \). Give \( K(\mathbb{Z}_2, n) \) a CW structure with \( n \)-skeleton the \( n \)-sphere. For notation purposes, elements \( \alpha \) of \( H^n(X) \) correspond to maps \( \alpha: X \to K(\mathbb{Z}_2, n) \).

The main thing we need is that if \( T \) is the transposition map on \( K(\mathbb{Z}_2, n) \wedge K(\mathbb{Z}_2, n) \), the there is a basepoint-preserving homotopy between the maps \( T \circ \iota \) and \( \iota \circ T \) mapping \( K(\mathbb{Z}_2, n) \wedge K(\mathbb{Z}_2, n) \to K(\mathbb{Z}_2, 2n) \). (Exercise: Construct this homotopy.) This is the same
thing as saying that $T^*(ι ⊗ t) = ι ⊗ t$. The homotopy between these two maps induces a map $Γ^1(K(ℤ_2, n)) → K(ℤ_2, 2n)$, which descends to a map $Λ^1K(ℤ_2, n) → K(ℤ_2, 2n)$. There is no obstruction to extending the resulting map to $λ: ΛK(ℤ_2, n) → K(ℤ_2, 2n)$. (Exercise: Prove this.) The pullback of the canonical class in $H^{2n}(K(ℤ_2, n))$ along $λ$ is the desired class $λ(ι)$. From this and naturality we get $λ(α)$ for arbitrary $α$.

Finally, consider the inclusion $RP^∞ × X ↪ Γ(X)$ coming from the diagonal map $X ↪ X × X$. We compose with the quotient map onto $ΛX$ to get a map $∇: RP^∞ × X → ΛX$. Then we have

$$∇^*: H^*(ΛX) → H^*(RP^∞) ⊗ H^*(X)$$

For any $α$ in $H^n(X)$, $∇^*(λ(α)) = ∑_{i=0}^n h^{n-i} ⊗ Sq^i(α)$; that is, the Steenrod squares are whatever make this true.