MTH 961: Suggested Exercises for Week 13

1. Let γ^n be the canonical n-plane bundle over $\mathrm{Gr}_n(\mathbb{R}^\infty)$. Prove that $\gamma^n \oplus \gamma^n$ is orientable and has $w_{2n}(\gamma^n \oplus \gamma^n) \neq 0$, hence $e(\gamma^n \oplus \gamma^n) \neq 0$. If n is odd, show that $2e(\gamma^n \oplus \gamma^n) = 0$.

2. Consider the complex Grassmannian $\mathrm{Gr}_n(\mathbb{C}^\infty)$. This has a canonical oriented $2n$-plane real bundle ξ^{2n} (by regarding each complex n-dimensional subspace as an oriented real n-dimensional subspace). Show the restriction of this bundle to the real subspace $\mathrm{Gr}_n(\mathbb{R}^\infty)$ is isomorphic to $\gamma^n \oplus \gamma^n$, and conclude that $e(\xi^{2n}) \neq 0$.

3. Let S^n be the unit sphere and $A \subset S^n \times S^n$ be the anti-diagonal, consisting of pairs of antipodal unit vectors. Using stereographic projection, show that TS^n is canonically homeomorphic to $S^n \times S^n - A$. Hence show that $H^*(E, E_0) \simeq H^*(S^n \times S^n, S^n \times S^n - \Delta) \simeq H^*(S^n \times S^n, A) \subset H^*(S^n \times S^n)$. Now, if n is even, show that $e(TS^n)$ is twice a generator of $H^n(S^n;\mathbb{Z})$. Conclude that TS^n possesses no nontrivial subbundles.

   [This is worked out in Hatcher's Vector Bundles book; you are encouraged to go read through it if you get stuck.]

4. Use last week's exercises to give an example of a bundle with vanishing Euler class but no nowhere zero section.

5. *A construction of the Steenrod squares.* Everything below is in $\mathbb{Z}_2$-coefficients. Notation mostly chosen to match Hatcher's Algebraic Topology Section 4L, although he does all the other Steenrod prime powers at the same time as the squares.

   In theory, a squaring operation would involve $X \times X$, but it's actually easier to work with $X \wedge X$, the smash product. This has a $\mathbb{Z}_2$ action generated by the map T (for transposition) that interchanges the factors; the basepoint x_0 in the smash product is a fixed point of the action. Consider the Borel construction

$$\Gamma X = (X \wedge X) \times_{\mathbb{Z}_2} S^\infty := ((X \wedge X) \times S^\infty)/((x_1, x_2), z) \sim (x_2, x_1, -z)).$$

   There is a fibre bundle $(X \wedge X) \hookrightarrow \Gamma X \xrightarrow{p} \mathbb{RP}^\infty$. Furthermore, since $x_0 \in X \times X$ is a fixed point, we have a basepoint section $\mathbb{RP}^\infty \hookrightarrow Y$. We let the quotient of ΓX by this copy of $\mathbb{RP}^\infty$ be ΛX, which is now a basepointed space. If we restrict this entire construction to S^1, we get subspaces $\Gamma^1 X$ and $\Lambda^1 X$. (Exercise: All of these constructions are natural, and if X has the structure of a CW complex, so do ΓX, ΛX, $\Gamma^1 X$, and $\Lambda^1 X$.)

   Now, there is an isomorphism

$$H^*(X \wedge X) \to H^*(X) \otimes H^*(X).$$

   (Exercise: Convince yourself this is true, if necessary.) In particular, we can think about $\alpha \otimes \alpha$ as an element of $H^{2n}(X \wedge X)$. Our goal is to construct an element $\lambda(\alpha) \in H^{2n}(\Lambda X)$ that restricts to $\alpha \otimes \alpha$ on each fibre $X \wedge X \subset \Lambda X$. By naturality, it suffices to construct a suitable $\lambda(\iota) \in H^{2n}(K(\mathbb{Z}_2, n))$, where ι is the fundamental class in $H^n(K(\mathbb{Z}_2, n))$. Give $K(\mathbb{Z}_2, n)$ a CW structure with n-skeleton the n-sphere. For notation purposes, elements α of $H^n(X)$ correspond to maps $\alpha\colon X \to K(\mathbb{Z}_2, n)$.

   The main thing we need is that if T is the transposition map on $K(\mathbb{Z}_2, n) \wedge K(\mathbb{Z}_2, n)$, the there is a basepoint-preserving homotopy between the maps $\iota \otimes \iota$ and $\iota \otimes \iota \circ T$ mapping $K(\mathbb{Z}_2, n) \wedge K(\mathbb{Z}_2, n) \to K(\mathbb{Z}_2, 2n)$. (Exercise: Construct this homotopy.) This is the same thing as saying that $T^*(\iota \otimes \iota) = \iota \otimes \iota$. The homotopy between these two maps induces a map $\Gamma^1(K(\mathbb{Z}_2, n)) \to K(\mathbb{Z}_2, 2n)$, which descends to a map $\Lambda^1 K(\mathbb{Z}_2, n) \to K(\mathbb{Z}_2, 2n)$. There is no obstruction to exending the resulting map to $\lambda\colon \Lambda K(\mathbb{Z}_2, n) \to K(\mathbb{Z}_2, 2n)$. (Exercise: Prove this.)The pullback of the canonical class in $H^{2n}(K(\mathbb{Z}_2, n))$ along λ is the desired class $\lambda(\iota)$. From this and naturality we get $\lambda(\alpha)$ for arbitrary α.

Finally, consider the inclusion $\mathbb{RP}^\infty \times X \hookrightarrow \Gamma(X)$ coming from the diagonal map $X \hookrightarrow X \times X$. We compose with the quotient map onto ΛX to get a map $\nabla\colon \mathbb{RP}^\infty \times X \to \Lambda X$. Then we have

$$\nabla^*\colon H^*(\Lambda X) \to H^*(\mathbb{RP}^\infty) \otimes H^*(X)$$

For any α in $H^n(X)$, $\nabla^*(\lambda(\alpha)) = \sum_{i=0}^n h^{n-i} \otimes Sq^i(\alpha)$; that is, the Steenrod squares are whatever make this true.