MTH 961: Suggested Exercises for Week 10

Depending on our progress today, there may be a new exercise sheet with exercises on vector bundles on Thursday.

1. Show that the construction of G-equivariant cohomology given in class satisfies the following property: If $f : X \to Y$ is a G-equivariant homotopy equivalence, then $H^*_G(X) \simeq H^*_G(Y)$.

2. Given a space X with an involution τ, show that the two constructions of the spectral sequence from $H^*(X;\mathbb{F}_2) \otimes \mathbb{F}_2[\theta]$ to $H^*_{\mathbb{Z}_2}(X;\mathbb{F}_2)$ give the same spectral sequence. (Hint: Consider a simplicial chain complex for $E\mathbb{Z}_2 = S^\infty$.)

3. Prove that given X as in the previous exercise, $H^*_{\mathbb{Z}_2}(X) = \mathrm{Ext}_{\mathbb{F}_2[\mathbb{Z}_2]}(C_*, \mathbb{F}_2)$, where $\mathbb{F}_2$ represents the trivial $\mathbb{F}_2[\mathbb{Z}_2]$-module.

4. Compute the equivariant cohomology of S^n with a reflection fixing S^k, for $k < n$.(Note that localization does not tell you everything about the equivariant cohomology.)

5. In class we found that given a space X with the homotopy type of a finite dimensional CW complex, and an involution τ on X, we had an isomorphism between $\theta^{-1}H^*_{\mathbb{Z}_2}(X)$ and $H^*(X^{\mathrm{fix}};\mathbb{F}_2) \otimes \mathbb{F}_2[\theta]$. This isomorphism can be rephrased as follows: Consider the natural map $H^*_{\mathbb{Z}_2}(X) \to H^*_{\mathbb{Z}_2}(X^{\mathrm{fix}})$. Then the kernel and cokernel of this map are both θ-torsion modules.

   It is possible to deduce this for general actions of a finite group G without spectral sequences using the fact that there is a Mayer-Vietoris sequence for equivariant cohomology (assuming that one decomposes a space into two equivariant neighborhoods). Work out the details. Hint: Finiteness of both G and the cell structure of X are both important to the argument.