Lecture 9

<u>Recall</u> <u>Bott Periodicity</u>

$$\begin{cases} U \simeq \Omega^2 U \\ O \simeq \Omega^8 O \\ Sp \simeq \Omega^8 Sp \end{cases} \qquad \begin{cases} O \simeq \Omega^4 Sp \\ Sp \simeq \Omega^4 O \end{cases}$$

This means you can construct a cohomology theory out of U or O.

$$\tilde{K}^{-i}(X) = \langle X, \Omega^{i-1} U\rangle \qquad \tilde{K}O^{i}(X) = \langle X, \Omega^{i-1} O\rangle$$

In particular $\tilde{K}^{even} = \langle X, BU\rangle$

$\tilde{K}^{odd} = \langle X, U\rangle$

<u>Method of Proof</u>

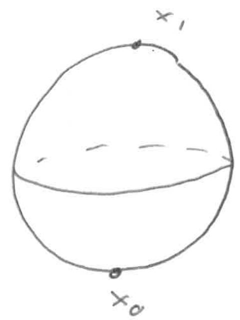
Morse Theory on $\mathcal{P}(p,q)$

$$\begin{aligned} \mathcal{P}(p,q) &\longrightarrow \mathbb{R} \\ \gamma &\longmapsto \text{length}^2(\gamma) \end{aligned} \quad \Bigg\} \quad \begin{array}{l} \text{Critical points are geodesics} \\ \text{Lowest index critical pts} \\ \text{are minimal geodesics.} \end{array}$$

<u>Bott</u> If the space $\mathcal{P}^d(p,q)$ of minimal geodesics is a mfd ; all other geodesics have index $\geq \lambda_0$, $\pi_i(\mathcal{P}(p,q), \mathcal{P}^d(p,q)) = 0$ for $0 \leq d \leq \lambda_0$. In particular $\pi_i(\mathcal{P}) \simeq \pi_{i+1}(M)$.

Example S^n

e.g. $S^3 = SU(2)$



- x_0, x_1 antipodal points
- Geodesics from x_0 to x_1
- Shortest geodesics S^{n-1}
- Geodesics wrapping $\frac{3}{2}$ times are index $2(n-1)$-cells

This is a good model of the general case: The space of minimal geodesics in $SU(2m)$ from I to $-I$ is $G_m(\mathbb{C}^{2m})$ and every nonminimal geodesic has index $\geq 2m+2$.

In particular, one shows $\pi_{i+1}(U(2m)) \cong \pi_i(G_m(\mathbb{C}^{2m}))$ for $i < 2m$. We also have

$$U(m) \longrightarrow V_m(\mathbb{C}^{2m}) \longrightarrow G_m(\mathbb{C}^{2m}) \quad \text{a fibre bundle.}$$

$$\pi_i(G_m(\mathbb{C}^{2m})) \simeq \pi_{i-1}(U(m)) \quad \text{for } m \gg i$$

$$\begin{array}{c} \| \wr \\ \pi_{i+1}(U(2m)) \end{array}$$

Hence $\pi_{i+1}(U) = \pi_{i+1}(U)$.

<u>What are these theories?</u> Eventually we will see that $\tilde{K}$ and $\widetilde{KO}$ are classes of <u>vector bundles</u> (fibre bundles w/ fibre a vector space) given a group structure.

Before going on, a sequence of spaces we haven't had yet.

__Claim__ Given a Fibration $F \to E \to B$, there is a "les" of spaces

$$\begin{tikzcd} \cdots \\ \Omega F \arrow[r] & \Omega E \arrow[r] & \Omega B \arrow[dll, bend left] \\ F \arrow[r] & E \arrow[r] & B \end{tikzcd}$$

Where any two maps form a fibration up to homotopy equivalence and any map after the first two is obtained by applying Ω to later maps.

__PF__ We iterate the process of forming homotopy fibres.

$$\begin{tikzcd} F_j \arrow[r] \arrow[d, shift left] & F_i \arrow[r, "i"] \arrow[d, shift right] & F_p \arrow[r] & E \arrow[r] \arrow[d, equal] & B \arrow[d, equal] \\ \Omega E \arrow[r] \arrow[u, shift left] & \Omega B \arrow[r] \arrow[u, shift right] & F \arrow[r, hook] \arrow[u, "s"] & E \arrow[r] & B \end{tikzcd}$$

__Recall__ $F_p = \{(e,\gamma) : e \in E,\ \gamma : I \to B,\ \gamma(0) = b_0,\ \gamma(1) = p(e)\}$

We have an inclusion $F = p^{-1}(b_0) \hookrightarrow F_p$ which is a homotopy equivalence.

$$e \longmapsto (e, c_{b_0})$$

Indeed we have an extension $F_p \to E$ which is a fibration

$$(e,\gamma) \longmapsto e$$

(in fact the pullback of
$$\begin{tikzcd} F_p \arrow[r] \arrow[d] & PB \arrow[d] \\ E \arrow[r] & B \end{tikzcd}$$
). So we can extend to

get a homotopy fiber F_i. The (actual) fiber of i over $e_0 \in p^{-1}(b_0)$ is pairs (e_0, γ) w/ γ a loop in B based at b_0, so just ΩB. As previously $\Omega B \hookrightarrow F_p$ is a homotopy equivalence. Note that $\Omega B \hookrightarrow F$ is the composition $\Omega B \hookrightarrow F_i \longrightarrow \underbrace{F_p \longrightarrow F}_{\text{homotopy}}$

so the square in the sequence containing these maps commutes up to homotopy. Now we can iterate...

<u>Corallary</u> For any K a CW cpx, there is a long exact sequence

$$\to \langle K, \Omega F\rangle \longrightarrow \langle K, \Omega E\rangle \longrightarrow \langle K, \Omega B\rangle \longrightarrow \langle K, F\rangle \longrightarrow \langle K, E\rangle \longrightarrow \langle K, B\rangle \longrightarrow \cdots$$

[Generalizes the les of a fibration]

Towers and obstructions

Given a space X, we have the following possible approximations of X.

n-skeleton X^n	$H_m(X^n) = 0$ for $m \geq n+1$	$H_k(X^n) \xrightarrow[\sim]{i_*} H_k(X)$ $k<n$ $H_n(X^n) \twoheadrightarrow H_n(X)$
Universal cover $\tilde{X}$	$\pi_1(\tilde{X}) = 0$	$\pi_k(\tilde{X}) \xrightarrow{\sim} \pi_k(X)$ for $k>1$.
n-connected cover $X^{(n)}$	$\pi_m(X^{(n)}) = 0$ $\forall$ $m<n$ $H_m(X^{(n)}) = 0$	$X^{(n)} \hookrightarrow X$ induces $\pi_k(X^{(n)}) \xrightarrow{\sim} \pi_k(X)$ $k>n$
Postnikov Tower	$\pi_m(X^n) = 0$ $m>n$	$X \to X_n$ induces $\pi_k(X) \simeq \pi_k(X_n)$ $k \leq n$

How would you get an n-connected cover?

Start w/ X, Add cells to X of dimension $(n+2)$ and higher to kill π_{n+1} and up. Take the homotopy fibre of the map $X \xrightarrow{f} Y$; then $F_f \hookrightarrow E_f \to Y$ gives us F_f as the n-connected cover.

More generally

Defn Given (X, A), we say (Z, A) is an n-connected approximation to (Z, A) if there is a map $f:(Z, A) \longrightarrow (X, A)$ inducing

$$\begin{cases} \pi_i(Z) \xrightarrow{\sim} \pi_i(X) & \text{for } i > n \\ \pi_i(A) \xrightarrow{\sim} \pi_i(Z) & \text{for } i < n \qquad \text{(i.e. } (Z, A) \text{ is n-connected)} \\ \pi_n(A) \twoheadrightarrow \pi_n(Z) \hookrightarrow \pi_n(X) \end{cases}$$

Propn Every (X, A) has an n-connected approximation.

Pf Almost identical to the proof that CW approximations exist.

Let $Z^n = A^n$.

- $Z^{n+1} = Z^n \cup ((n+1)\text{-cells})$. Attach cells to ① make $\pi_n(Z) \hookrightarrow \pi_n(X)$
  ② make $\pi_{n+1}(Z)$ generate $\pi_{n+1}(X)$

- $Z^{n+2} = Z^{n+1} \cup ((n+2)\text{-cells})$. Attach $(n+2)$-cells to ① make $\pi_{n+1}(Z) \cong \pi_{n+1}(X)$
  ② make $\pi_{n+2}(Z)$ generate $\pi_{n+1}(X)$

More generally: given an arbitrary $X \xrightarrow{f} Y$, we can find Z_n an n-connected approximation to X such that

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowleft"] & & Y \arrow[d] \\ X \arrow[r, "g"'] & Z_n \arrow[r, "h"'] & Y \end{tikzcd}$$

and $h_* : \pi_i(Z) \xrightarrow{\sim} \pi_i(Y)$ for $i > n$

$g_* : \pi_i(X) \xrightarrow{\sim} \pi_i(Z)$ for $i < n$

$\pi_n(X) \twoheadrightarrow \pi_n(Z) \hookrightarrow \pi_n(Y)$

[Reduces to the previous case by considering $X \hookrightarrow M_f \simeq Y$]

<u>Moreover</u> If $n \geq n'$ and $\{Z_n, Z_{n'}\}$ are $\{n, n'\}$-connected approximations to f, we have a map

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & Z_n \arrow[r] \arrow[d] & Y \\ & Z_{n'} \arrow[ur] & \end{tikzcd}$$

commuting up to homotopy

<u>Example</u> Whitehead Towers

$$\begin{tikzcd} f : \{pt\} \arrow[r] & X \\ Z_0 \arrow[u] \arrow[ur] & \\ Z_1 \arrow[u] \arrow[uur] & \\ Z_2 \arrow[u] \arrow[uuur] & \\ Z_3 \arrow[u] \arrow[uuuur] & \end{tikzcd}$$

Z_0 = connected cpt of the basepoint in X

$Z_1 = \tilde{X}$.

$\vdots$

<u>Postnikov Towers</u> $f : X \to \{pt\}$

$$\begin{tikzcd}
 & \vdots \arrow[d] & \\
 & Z_2 \arrow[d] \arrow[ddr] & \\
 & Z_1 \arrow[d] \arrow[dr] & \\
X \arrow[uur] \arrow[ur] \arrow[r] & Z_0 \arrow[r] & \{pt\}
\end{tikzcd}$$

$Z_0 = \{pt\}$

$Z_1 = \{pt\}$

$Z_2 = K(\pi_1(X), 1)$

$\vdots$

$$\pi_n(X) \twoheadrightarrow \overset{0}{\overset{\cong}{\pi_n(Z_n)}} \hookrightarrow \pi_n(pt)$$

$$\begin{cases} \pi_k(Z_2) = 0 & k \geq 2 \\ \pi_k(Z_2) = \pi_1(X) & \ldots \end{cases}$$

In fact we claim this is a tower of fibrations, which we will prove next time.