Lecture 8

Recall **Goal** Understand an example of an extraordinary cohomology theory.

Last Time Given $F \hookrightarrow E \to B$ Fibration w/ E contractible, $\exists$ whe

$$\begin{array}{ccccc} F & \hookrightarrow & E & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ \Omega B & \hookrightarrow & PB & \longrightarrow & B \end{array}$$

For example, we have $O(k) \longrightarrow V_k(\mathbb{R}^\infty) \longrightarrow G_k(\mathbb{R}^\infty)$

($V_k(\mathbb{R}^\infty)$ contractible)

Implies that $\pi_{i-1}(G_k(\mathbb{R}^\infty)) \xrightarrow{\sim} \pi_i(O(k))$ so $O(k) \simeq \Omega G_k(\mathbb{R}^\infty)$.

For example, $O(1) \simeq \Omega(\mathbb{RP}^\infty)$ $\qquad$ $U(1) \simeq \Omega \mathbb{CP}^\infty$

$$S^0 = O(1), \quad \Omega(\mathbb{RP}^\infty) = \Omega(K(\mathbb{Z}_2,1)), \quad S^0 \simeq \Omega(K(\mathbb{Z}_2,1))$$

$$S^1 = U(1), \quad \Omega\mathbb{CP}^\infty = \Omega(K(\mathbb{Z},1)), \quad S^1 \simeq \Omega(K(\mathbb{Z},1))$$

In general For any topological group, can find EG contractible on which G acts freely.

$$G \longrightarrow EG \longrightarrow BG \quad \Big\} \text{ Fibration}$$

(EG contractible)

So that $G \simeq \Omega BG$. BG is the <u>classifying space</u> for G.

Example: What is the classifying space of a discrete group G? $BG = K(G,1)$

Explicit Construction of EG: For G discrete:

Take a union: $[g_0] \cup \{[g_0, g_1]\} \cup \{[g_0, g_1, g_2]\} \cup \dots \cup \{[g_0, \dots, g_n]\}$

$[g_0]$ ↑ A zero-simplex for every element of G

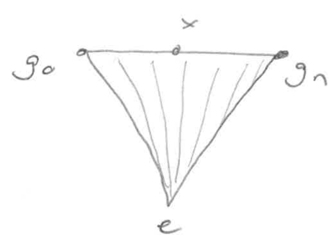
$\{[g_0, g_1]\}$ ↑ A 1-simplex for every pair of elements

$$\partial([g_0, \dots g_n]) = \bigcup_{i=0}^{n} (-1)^i [g_0 \dots \hat{g_i} \dots g_n]$$

This is

(1) Contractible: For $x \in EG$, $x \in [g_0, \dots, g_n] \in \partial[e, g_0, \dots, g_n]$

Contract to e along a straight line homotopy.



(2) G acts freely $g[g_0, \dots, g_n] = [gg_0, \dots, gg_n]$

(3) Corresponding $BG = EG/G = \{\cup [e, g_1, \dots, g_n]\}$

w/ $\partial([e, g_1, \dots, g_n]) = [g_1, \dots, g_n] \cup \left(\bigcup_{i=1}^{n} (-1)^i [e, g_2, \dots, \hat{g_i}, g_n]\right)$

$= [e, g_1^{-1} g_2, \dots, g_1^{-1} g_n] \cup (\quad = \quad)$

Or more typically BG is

$$\{[h_1|\dots|h_n] = [1, h_1, h_1h_2, \dots, h_1\cdots h_n]\}$$ (Bar complex for G

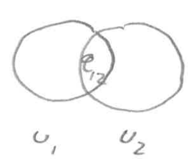
$$\partial[h_1|\dots|h_n] = [h_2|h_3|\dots|h_n] \cup -[h_1h_2|\dots|h_n] \cup \dots$$
$$\cup\, [h_1|\dots|h_jh_{j+1}|\dots|h_n] \cup \dots \cup \pm[h_1|\dots|h_{n-1}]$$

Can check this is a $K(G,1)$.

Why the name "classifying space"?

Defn A G-bundle over X is a fibre bundle $E \downarrow X$ w/ Fiber G and over lap maps given by $g_{12}: U_1 \cap U_2 \to G$ such that $e_{12}(x,g) = (x, g_{12}(x)g)$.

$(x,g) \in U_1 \times G$, $(x, g_{12}(x)g) \in U_2 \times G$



[IF necessary: A fibre bundle is a fibration $E \xrightarrow{p} B$ such that for every $x \in B$, $\exists$ a nbhd U of x st $p^{-1}(U) \overset{h}{\Rightarrow} U \times F$ is homeomorphic to a product. The map h is called a local trivialization.]

Propn For every G-bundle $E \to X$, we have $F : X \to BG$ st

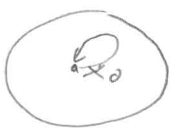
$$\begin{array}{ccc} E \simeq F^*(EG) & \longrightarrow & EG \\ \downarrow \swarrow & & \downarrow p \\ X & \xrightarrow{F} & BG \end{array}$$

$$E = F^*(EG) = \{ (x,y) : x \in X,\ y \in EG \text{ w/ } p(y) = F(x) \}$$

Pf True in general, but right now we have the tools to see it for G discrete. We have a holonomy map $\pi_1(X) \to G$

Then $(G\text{-bundles}) \longleftrightarrow (\text{maps } \pi_1(X) \to \mathrm{Aut}(F_{x_0}))$

↑ via holonomy map



To see holonomy is surjective, take $(\hat{X} \times G)/\pi_1(X)$ so that $\pi_1(X)$ acts on G by e. So

$$(G\text{-bundles}) \longleftrightarrow (\text{maps } \pi_1(X) \to \mathrm{Aut}(F(x_0))) \longleftrightarrow \text{maps } (X, K(G,1))$$

↑ modulo conjugation

Back to Fibre bundles

$$\begin{array}{ccc} O(k) & \longrightarrow & V_k(\mathbb{R}^n) \\ & & \downarrow \\ & & G_k(\mathbb{R}^n) \end{array} \qquad \begin{array}{ccc} V_{k-e}(\mathbb{R}^{n-e}) & \longrightarrow & V_k(\mathbb{R}^n) \\ & & \downarrow \\ & & V_e(\mathbb{R}^n) \end{array}$$

Set $k=n$: $O(k-\ell) \longrightarrow O(k)$

$$\downarrow$$

$$V_\ell(\mathbb{R}^k)$$

If $\ell=1$, have $O(k-1) \longrightarrow O(k)$

$$\downarrow$$

$$V_1(\mathbb{R}^k) = S^{k-1}$$

$\Rightarrow \pi_i(O(k)) \cong \pi_i(O(k-1))$ for $i < k-2$. So we have $O(1) \hookrightarrow O(2) \hookrightarrow O(3) \hookrightarrow \ldots$ all of which are isomorphisms on homotopy groups. Union of these is $O(\infty)$ or O.

Similarly, define $\begin{cases} U(\infty) = U \\ Sp(\infty) = Sp \end{cases}$

Note reminder

$Sp(n) = \{ A \in GL(n, \mathbb{H}) : A^*A = I \}$

Theorem (Bott Periodicity) $\begin{cases} U = \Omega^2 U \\ O = \Omega^8 O \\ Sp = \Omega^8 Sp \end{cases}$ $\begin{cases} O = \Omega^4 Sp \\ Sp = \Omega^4 O \end{cases}$

This makes computing the homotopy groups of U, O, Sp not too bad.

π_i	0	1	2	3	4	5	6	7
O	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
U	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
Sp	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$

Note Can check $\pi_1(O(2)) = \mathbb{Z}$

$\pi_1(O(3)) = \mathbb{Z}_2$ ← stabilizes here

Corollary $X \mapsto \langle X, O \rangle$ define cohomology theories, called real
$X \mapsto \langle X, U \rangle$ and complex K-theory.

Here $\widetilde{KO}^{-i}(X) = \langle X, \Omega^i O \rangle$ } 8-periodic

$\widetilde{K}^{-i}(X) = \langle X, \Omega^i U \rangle$ } 2-periodic

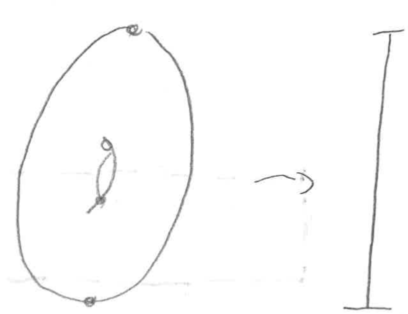
Notes on Bott's proof

Want to build a finite-dimensional model for ΩU. Idea is to look at paths on some Riemannian manifold X w/ fixed endpoints $\mathcal{P}(p,q)$.

$\mathcal{P}(p,q) \longrightarrow \mathbb{R}$ is a Morse Ftn on this space.
$\gamma \longmapsto \text{length}^2(\gamma)$ Critical points are geodesics.

Morse Ftn ⇒ CW cpx. Index i critical pts are i-dim'l cells.

[Example

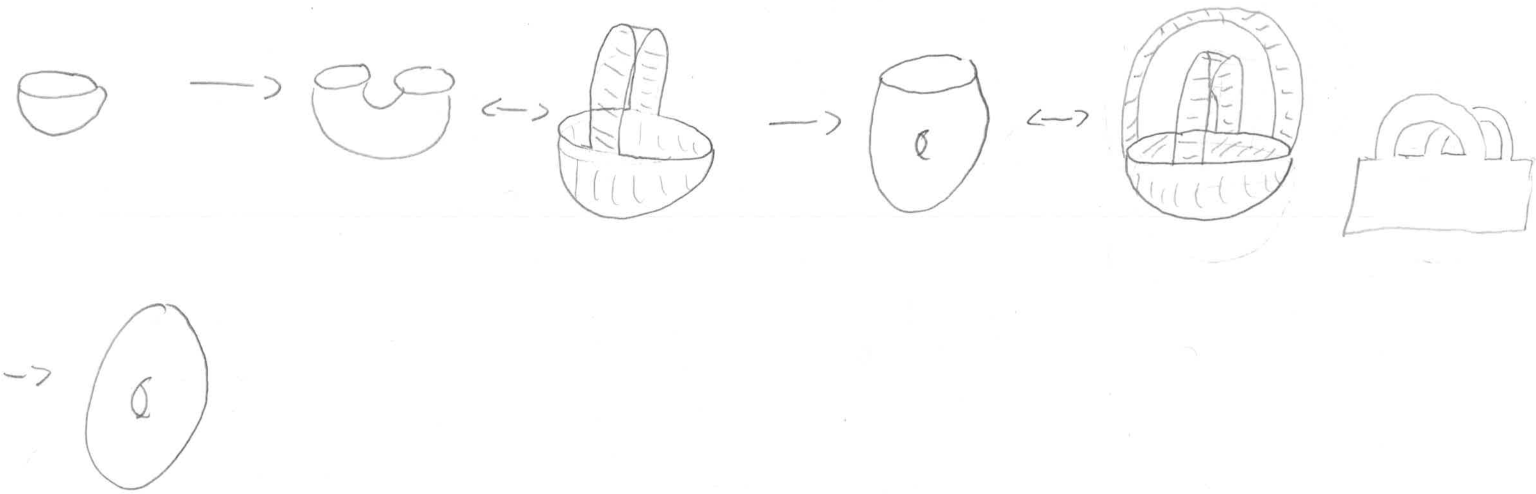


A Morse ftn is a smooth function $M \to \mathbb{R}$ w/ isolated critical pts. Near any critical pt, the function can be modelled as

$$F(x_1, \ldots, x_m) = x_m^2 + \cdots + x_{n+1}^2 - x_n^2 - \cdots - x_1^2$$

The <u>index</u> of the critical pt is n (the number of linearly independent directions in which the function is decreasing).

Critical pt of index $i \longleftrightarrow$ cell of dimn i.



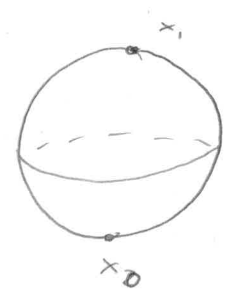
We can do this with the path space.

<u>Bott</u> If the space $P^d(p,q)$ of minimal geodesics is a mfld & all other geodesics have index $\geq \lambda_0$, $\pi_i(P, P^d) = 0$ for $0 \leq d \leq \lambda_0$.

In particular $\pi_i(P) \simeq \pi_{i+1}(M)$.

<u>Example</u> S^n

e.g. $S^3 = SU(2)$



- x_0, x_1 antipodal points
- Geodesics from x_0 to x_1
- Shortest geodesics $\simeq S^{n-1}$
- Geodesics wrapping 3/2 times are index $2(n-1)$-cells

This is a good model: The space of minimal geodesics in $SU(2m)$ from I to $-I$ is $G_m(\mathbb{C}^{2m})$ and every nontrivial geodesic has index $\geq 2m+2$.

In particular, one shows

$$\pi_{i+1}(U(2m)) \cong \pi_i(G_m(\mathbb{C}^{2m}))$$ For $i < 2m$. We also

have $U(m) \longrightarrow V_m(\mathbb{C}^{2m}) \longrightarrow G_m(\mathbb{C}^{2m})$ Fibre bundle.

$$\pi_i(G_m(\mathbb{C}^{2m})) \simeq \pi_{i-1}(U(m)) \quad \text{For } m \gg i$$

$$\cong$$

$$\pi_{i+1}(U(2m))$$

Hence $\pi_{i-1}(U) = \pi_{i+1}(U)$.