Lecture 7

Recall Goal: If $\tilde h^n(X)$ is a (reduced) cohomology theory and $\tilde h^n(S^0) = 0$ for $n \neq 0$, then $\tilde h^n(X) = \tilde H^n(X; h^0(pt))$.

Idea: Show

$$\begin{tikzcd}[column sep=small]
& h^{n-1}(X^{n-1}, X^{n-2}) \arrow[rr] \arrow[dr] & & h^n(X^n, X^{n-1}) \arrow[rr] \arrow[dr] & & h^{n+1}(X^{n+1}, X^n) \arrow[dr] & \\
h^{n-2}(X^{n-2}) \arrow[ur] & & h^{n-1}(X^{n-1}) \arrow[ur] & & h^n(X^n) \arrow[ur] & & {}
\end{tikzcd}$$

is the same as the complex for cellular cohomology.

Left to show: Differentials are degree maps

$$\begin{tikzcd}
& G \arrow[d, phantom, sloped, "\cong"] & G \arrow[d, phantom, sloped, "\cong"] \\
h^n(D^n_\alpha, \partial D^n_\alpha) & \tilde h^{n-1}(\partial D^n_\alpha) \arrow[l, "\sim", "\partial"'] & \tilde h^{n-1}(S^{n-1}_\beta) \arrow[l, "\text{Attaching map}"'] \arrow[d, "\rho_\beta^*"] \arrow[dddl, bend left=70, "\rho_\beta^*"] \\
\Pi G \cong h^n(X^n, X^{n-1}) \arrow[u, "\text{Characteristic map}"] & \tilde h^{n-1}(X^{n-1}) \arrow[l, "\delta_n"'] \arrow[u] & \tilde h^{n-1}(X^{n-1}, X^{n-2}) \arrow[l] \\
& h^{n-1}(X^{n-1}, X^{n-2}) \arrow[ul, "\partial_n"] \arrow[u] \arrow[d, phantom, sloped, "\cong"] & h^{n-1}(X^{n-1}/X^{n-2}, X^{n-2}/X^{n-2}) \arrow[l, "\sim"'] \arrow[u, "\cong"'] \\
& \Pi G &
\end{tikzcd}$$

We need to check our notion of "degree" $F^*: h^n(S^n) \to h^n(S^n)$ accords w/ the ordinary one. This is clear for 0 and 1.

But any map $S^n \to S^n$ is a multiple of the identity, so since we checked last time that $(f+g)^* = f^* + g^*$, our notions of degree must match.

What is this good for?

(1) Universal relations on H^*

e.g. $H^1(X) = \langle X, S^1 \rangle$ for α a generator of $H^1(S^1; \mathbb{Z})$.

$f^*(\alpha) \leftarrow\!\shortmid f$

$\alpha^2 = 0 \Rightarrow$ for any $\beta \in H^1(X)$, $\beta^2 = 0$

Compare $H^1(X; \mathbb{Z}_2) = \langle X, \mathbb{RP}^\infty \rangle$ for α a generator of $H^1(\mathbb{RP}^\infty; \mathbb{Z}_2)$

$f^*(\alpha) \leftarrow\!\shortmid f$

(2) Cohomology operations

Given $\beta \in H^m(K(G,n); G')$ and $H^n(X; G) \longleftrightarrow \langle X, K(G,n) \rangle$

$\alpha \longleftrightarrow f$

Pull back β to get $f^*\beta \in H^m(X; G')$. Get a natural map

$H^n(X; G) \longrightarrow H^m(X; G')$

$\alpha \longmapsto f^*(\beta)$

(Important to understanding $H^*(K(G,n))$.)

Same thing for multiple inputs (e.g. cup product).

$$H^n(X;R) * H^m(X;R) \longrightarrow H^{n+m}(X;R)$$

$$\begin{tikzcd}
\langle X, K(R,n)\rangle \arrow[dr] & & \langle X, K(R,m)\rangle \arrow[dl] & H^{n+m}(X;R) \\
& \langle X, K(R,m) \wedge K(R,m)\rangle \arrow[urr, bend right] & &
\end{tikzcd}$$

So $H^{n+m}(\underbrace{K(R,n) \wedge K(R,m)}_{(n+m-1)\text{-ctd}})$ should have a canonical generator

$$\begin{aligned}
H^{n+m}(K(R,n) \wedge K(R,m)) &\simeq \mathrm{Hom}(H_{n+m}(K(R,n) \wedge K(R,m)), R) \\
&\simeq \mathrm{Hom}(\pi_{n+m}(\text{———}), R) \\
&\simeq \mathrm{Hom}(R \otimes R, R)
\end{aligned}$$

canonical map is multiplication.

<u>Example</u> The <u>Steenrod squares</u> are a set of cohomology operations

$$Sq^i : H^k(X;\mathbb{Z}_2) \longrightarrow H^{k+i}(X;\mathbb{Z}_2)$$

(so called b/c $Sq^k(\underset{\text{degree } k}{x}) = x \cup x$.)

Example $K(\mathbb{Z},n) = S^n \cup (n+2$ and higher cells$)$

$$\begin{array}{ccc} S^n \wedge S^m & \simeq & S^{n+m} \\ \text{switch} \downarrow & & \downarrow \varepsilon \\ S^m \wedge S^n & \simeq & S^{n+m} \end{array}$$

ε has degree $(-1)^{nm}$

$\Rightarrow \alpha \wedge \beta = (-1)^{\deg\alpha \deg\beta} \beta \wedge \alpha$

Other Fibre Bundles and Cohomology Theories

Frame Spaces and Grassmanians

Spaces over $\mathbb{R}$

1. $O(n)$
2. $G_k(\mathbb{R}^n) = \{ W \subseteq \mathbb{R}^n : W$ is a k-dim'l subspace $\}$
3. $V_k(\mathbb{R}^n) = \{$ ordered sets of orthonormal vectors $\}$] Frame space
4. $\mathbb{RP}^n \simeq G_1(\mathbb{R}^{n+1})$
5. $S^n \simeq V_1(\mathbb{R}^{n+1})$
6. $V_k'(\mathbb{R}^n) = \{$ ordered sets of k lin in dep vectors $\}$

Over $\mathbb{C}$

1. $U(n)$
2. $G_k(\mathbb{C}^n)$
3. $\mathbb{CP}^n = G_1(\mathbb{C}^{n+1})$
4. $V_k(\mathbb{C}^n)$
5. $S^{2n-1} = V_1(\mathbb{C}^n)$

Over $\mathbb{H}$

Note definitions vary!

1. $Sp(n)$
2. $G_k(\mathbb{H}^n)$
3. $\mathbb{H}P^n = G_1(\mathbb{H}^{n+1})$
4. $V_k(\mathbb{H})$
5. $S^{4n-1} = V_1(\mathbb{H}^n)$

Relations (Also $\mathbb{C}$, $\mathbb{H}$)

- $O(n) = V_n(\mathbb{R}^n)$

- $O(k) \longrightarrow V_k(\mathbb{R}^n) \longrightarrow G_k(\mathbb{R}^n)$ } This quotient map is how the topology of $G_k(\mathbb{R}^n)$ is defined.

- $O(n-k) \longrightarrow O(n) \longrightarrow V_k(\mathbb{R}^n) \qquad O(n)/O(n-k) = V_k(\mathbb{R}^n)$

- $O(n-k) \times O(k) \longrightarrow O(n) \longrightarrow G_k(\mathbb{R}^n) \qquad O(n)/O(n-k)\times O(k) \simeq G_k(\mathbb{R}^n)$

- $V_{k-\ell}(\mathbb{R}^{n-\ell}) \longrightarrow V_k(\mathbb{R}^n) \longrightarrow V_\ell(\mathbb{R}^n) \qquad V_\ell(\mathbb{R}^n) \simeq V_k(\mathbb{R}^n)/V_{k-\ell}(\mathbb{R}^{n-\ell})$

Lemma. $V_k(\mathbb{R}^n)$ is $((n-k+1)-2)$-connected

- $V_k(\mathbb{C}^n)$ is $(2(n-k+1)-2)$ connected
- $V_k(\mathbb{H}^n)$ is $(4(n-k+1)-2)$ - "
- $V_k(-^{\infty})$ is contractible.

a map into the loop space on F. By the Five lemma, $F \to \Omega B$ is a whe. $\square$

So we have $O(k) \longrightarrow V_k(\mathbb{R}^\infty) \longrightarrow G_k(\mathbb{R}^\infty)$.

contractible

Implies that $\pi_{i-1}(G_k(\mathbb{R}^\infty)) \xrightarrow{\sim} \pi_i(O(k))$, so $O(k) \simeq \Omega G_k(\mathbb{R}^\infty)$.

For example, $O(1) \simeq \Omega(\mathbb{R}P^\infty)$ $\qquad$ $U(1) \simeq \Omega \mathbb{C}P^\infty$

$$\begin{array}{ccc} O(1) & \simeq & \Omega(\mathbb{R}P^\infty) \\ \| & & \| \\ S^0 & \simeq & \Omega(K(\mathbb{Z}/2,1)) \end{array} \qquad \begin{array}{c} U(1) \\ \| \\ S^1 \end{array}$$

<u>In general</u> For any topological group, can find EG contractible on which G acts freely.

$G \longrightarrow EG \longrightarrow BG$ } fibration

contractible

So that $G \simeq \Omega BG$. BG is the <u>classifying space</u> for G.

<u>Example</u> What is the classifying space of a discrete group G?

$BG = K(G,1)$.