Recall from last time, given a CW pair \((X, A)\)

There exists \(A \longrightarrow X \longrightarrow X/A \longrightarrow \Sigma A \longrightarrow \Sigma X \longrightarrow \Sigma(X/A) \longrightarrow \Sigma^2A\)

so every three successive terms are homotopy equivalent to an inclusion/quotient triple.

\textbf{Claim} From this sequence of maps, get maps on homotopy classes of maps \(\langle x_1, x_2 \rangle\) which are exact.

\[ A \longrightarrow x \longrightarrow X/A \longrightarrow \Sigma A \]

\[ \langle A, k \rangle \longrightarrow \langle x, k \rangle \longrightarrow \langle X/A, k \rangle \longrightarrow \langle \Sigma A, k \rangle \]

\textbf{Pf} suffices to check for first three terms.

\textbf{Claim} \(\text{ker}(\varphi_1) = \text{im}(\varphi_2)\)

\[ \text{ker}(\varphi_1) = \text{maps } f : X \longrightarrow k \text{ at } f|_A : A \longrightarrow k \text{ is nullhomotopic} \]

\[ \text{im}(\varphi_2) = \langle x/A, k \rangle = \langle x \cup C.A, k \rangle \text{, or maps } \longrightarrow k \text{ that extend over } C.A \text{, this is the same thing}. \]

\textbf{Thm} \(\langle x, k_n \rangle\) is a cohomology theory for \(k_n\) an \(\Omega\)-spectrum.
To show les on cohomology,

\[ \tilde{h}^n(x/A) \rightarrow \tilde{h}^n(x) \rightarrow \tilde{h}^n(A) \rightarrow \tilde{h}^{n+1}(x/A) \rightarrow \tilde{h}^{n+1}(x) \]

This is exact as maps of sets, hence of abelian groups. Want

\[ \delta: \tilde{h}^n(A) \rightarrow \tilde{h}^n(x/A) \to be natural in (x,A) \rightarrow (y,B) \text{ cw maps}. \]

But we made no choices in defining it.

Now we want to check we get the expected thing for \( \tilde{h}(\mathbb{S}^n) \).

**Thm:** Let \( h^n \) be a cohomology theory on cw-epxes, \( \tilde{h}^n(pt) = 0 \) for \( h \neq 0 \). Then \( \tilde{h}^n(x,A) = H^n(x,A; h^0(pt)) \)

**Note:** Reduced \( \rightarrow \) unreduced

Let \( x_t = x \cup \mathbb{S}^n \) disjoint basepoint \( y \)

Set \( h^n(x) = \tilde{h}^n(x_t) \) is an unreduced theory.

**PF** Recall cellular cohomology \( h^n(pt) = G \).

\[ \rightarrow \prod_{(n-1)-\text{cells}} G \rightarrow \prod_{n-\text{cells}} G \rightarrow \prod_{(n+1)-\text{cells}} G \rightarrow \cdots \]

\[ H^n_{x_t}(x, G) \rightarrow H^n(x, G) \rightarrow H^n_{x_t}(x, G) \rightarrow \cdots \]
Goal: Show $h^n$ has the same complex.

PF: Start w/ the following picture.

Arrows of the same color come from a leg of a pair and compose to zero.

- $\partial^2 = 0$ since we pass through one of these compositions.

- $h^k(x^n, x^{n-1}) = \tilde{h}^k(x^n) = \tilde{h}^k(V, s \alpha) = \prod_{\alpha} h^k(s \alpha) = \sum_{G \alpha = n} \prod_{\alpha} G$ otherwise.

- From legs for $(x^n, x^{n-1})$, $h^k(x^n) = h^k(x^{n-1})$ if $k < n, n-1$.

- In particular, for $k > n$, $h^k(x^n) = h^k(x^n) = 0$.

Now let's look at this leg in more detail.
\[ \rightarrow h^{n-1}(x^{n-1}, x^{n-2}) \xrightarrow{\partial_{n-1}} h^n(x^n, x^{n-1}) \xrightarrow{\partial_n} h^n(x^n, x^n) \rightarrow \]

\[ h^{n-2}(x^{n-2}) \]

\[ h^{n-1}(x^n) \]

\[ h^n(x^n) \]

\[ 0 = h^{n-1}(x^{n-1}) \]

\[ h^n(x^n) \]

\[ 0 = h^n(x^n, x^n) \]

- \( h^n(x) = \ker(\partial_n) \)

- \( \text{Im}(\partial_n) = \text{Im}(\partial_n) \)

- \( \ker(\partial_n)/\text{Im}(\partial_{n-1}) = \ker(\partial_n)/\text{Im}(\partial_{n-1}) = \ker(\partial_n) = h^n(x) \)

Note: Have not yet justified the assertion that \( h^k(x^n) \cong h^k(x) \) for \( k < n \) yet.
We almost know the homology of the les is the expected thing. Still need to check: $h^k(x^n) = h^k(x)$ for $k \leq n$.

\[
\text{Mapping telescope } \bigcup_{i=k}^{\infty} x^i \xrightarrow{\ell^i} \infty \xrightarrow{\ell^i} \bigcup_{i=k}^{\infty} x^i
\]

Let $Z = \left( \bigcup_{i=k}^{\infty} x^i \times \mathbb{Z} \right) \cup \left( \mathbb{Z} \times \bigcup_{i=k}^{\infty} x^i \right)$ so $Z \cong \bigcup_{i=k}^{\infty} x^i$.

$T/Z = \left( \bigcup_{i=k}^{\infty} x^i \right) / (co)homology$ of $Z$ and $T/Z$ are shifted by a dimension.

\[
\begin{array}{cccccc}
& h^{k+1}(T, Z) & \xleftarrow{(\ell^i) \ast} & h^k(T) & \xleftarrow{\ell^i} & h^k(T, Z) & \xleftarrow{\ell^i} & h^{k-1}(Z) \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\prod h^{k+1}(X_i) & \xleftarrow{\ell^i} & \prod h^k(x^i) & \xleftarrow{\ell^i} & \prod h^k(T, X_i) & \xleftarrow{\ell^i} & \prod h^{k-1}(X_i)
\end{array}
\]

\[
\begin{pmatrix}
(1, x, 1, x, 1, x, \ldots) \\
\downarrow
\end{pmatrix}
\]

Consider the les.

For the pair
\[
(x \times I, x \times \Delta^0 \oplus x \times \mathbb{Z})
\]

shows this is surjective.

Inverse limit stabilizes since the groups $H^k(x^n)$ do as $n \to \infty$. 
Finally need to show $\partial_n$ are degree maps.

\[
\begin{align*}
\tilde{h}(p^n, \partial)p^n &\xrightarrow{\partial} \tilde{h}(\partial)p^n &\tilde{h}(p^n, \partial)p^n &\xrightarrow{\partial} \tilde{h}(\partial)p^n \\
\tilde{h}(\partial)p^n &\xrightarrow{e_\partial} \tilde{h}(\partial)p^n &\tilde{h}(\partial)p^n &\xrightarrow{e_\partial} \tilde{h}(\partial)p^n \\
\end{align*}
\]

We need to check our notion of "degree" $f^* : h^n(S^n) \to h^n(S^n)$ matches the ordinary one. This is clear for $G_1$. Any map $s^n \to S^n$ is a multiple of the identity, so it's a multiple of id $G_n$. Need to check additivity:

**Prop.** For any cohomology theory, given $f, g : X \to K$, have $(f + g)^* = f^* + g^*$.

\[
\begin{align*}
\tilde{h}(\Sigma X) &\xrightarrow{i^*} \tilde{h}(\Sigma X) &\tilde{h}(\Sigma X) &\xrightarrow{(f \circ g)^*} \tilde{h}(K) \\
\end{align*}
\]
\( h^c(x) \) and \( H^c_\bullet(x; G) \) have the same cellular chain cpx.
\[ \Rightarrow h^c(x) \cong H^c_\bullet(x; G). \]
One can check this is natural.

What is this good for?

1. Universal relations on \( H^c_\bullet \)

\[ \text{e.g. } H^1(x) = \langle x, s^1 \rangle, \text{ for } \alpha \text{ a generator of } H^1(S^1; \mathbb{Z}), \]
\[ F^c(\alpha) \leftarrow f \]
\[ \alpha^2 = 0 \Rightarrow \text{for any } B \in H^1(x), B^2 = 0. \]

Compare \( H^1(x; \mathbb{Z}_2) = \langle x, \text{IRIP}^\infty \rangle \) for \( \alpha \) a generator of \( H^1(\text{IRIP}^\infty; \mathbb{Z}_2) \)

\[ F^c(\alpha) \leftarrow f \]

2. Cohomology Operations

Given \( B \in H^m(K(G_n); G') \) and \( H^n(x; G) \leftarrow \langle x, \kappa(G_n) \rangle \)
\[ \Rightarrow c \rightarrow f^c \]

Pull back \( B \) to get \( F^c B \in H^m(x; G') \). Get a natural map

\[ H^n(x; G) \rightarrow H^m(x; G') \]
\[ \alpha \rightarrow f^c(\alpha). \]

(Very important to understanding \( H^c(K(G_n)) \),