Lecture 6

Recall from last time, given a CW pair (X, A)

$\exists$ an "les" $A \hookrightarrow X \to X/A \to \Sigma A \to \Sigma X \to \Sigma(X/A) \to \Sigma^2 A$

st every three successive terms are htpy equivalent to an inclusion/quotient triple.

Claim From this sequence of maps, get maps on homotopy classes of maps $\langle\ , K\rangle$ which are exact.

$$A \hookrightarrow X \to X/A \to \Sigma A$$

$$\langle A, K\rangle \xleftarrow{\varphi_1} \langle X, K\rangle \xleftarrow{\varphi_2} \langle X/A, K\rangle \leftarrow \langle \Sigma A, K\rangle$$

PF suffices to check for first three terms.

Claim $\ker(\varphi_1) = \operatorname{im}(\varphi_2)$

$\ker(\varphi_1) = \{\text{maps } f : X \to K \text{ st } f|_A : A \to K \text{ is nulhomotopic}\}$

$\operatorname{im}(\varphi_2) = \langle X/A, K\rangle = \langle X \cup C.A, K\rangle$, or maps $X \to K$ that extend over $C.A$. This is the same thing.

Thm $\langle X, K_n\rangle$ is a cohomology theory for K_n an Ω-spectrum.

PF To show les on cohomology,

$$\begin{tikzcd}
\tilde h^n(X/A) \arrow[r] \arrow[d, equal] & \tilde h^n(X) \arrow[r] \arrow[d, equal] & \tilde h^n(A) \arrow[r] \arrow[d, equal] & \tilde h^{n+1}(X/A) \arrow[r] \arrow[d, equal] & \tilde h^{n+1}(X) \\
\langle \Sigma(X/A), K_{n+1}\rangle \arrow[r] & \langle \Sigma X, K_{n+1}\rangle \arrow[ddr, bend right] & \langle A, K_n\rangle \arrow[d, equal] & \langle X/A, K_{n+1}\rangle \arrow[r] & \langle X, K_{n+1}\rangle \\
& & \langle A, \Omega K_{n+1}\rangle \arrow[d, equal] & & \\
& & \langle \Sigma A, K_{n+1}\rangle \arrow[uur, bend right, "\delta"'] & &
\end{tikzcd}$$

This is exact as maps of sets; hence of abelian groups. Want $\delta : \tilde h^n(A) \to \tilde h_n(X/A)$ to be natural in $(X, A) \to (Y, B)$ cw maps.

But we made no choices in defining it.

Now we want to check we get the expected thing for $K(G,n)$.

Thm Let h^n be a cohomology theory on cw-cpxes, $\tilde h^n(pt) = 0$ for $n \neq 0$. Then $\tilde h^n(X, A) = H^n(X, A; h^0(pt))$

Note Reduced $\longleftrightarrow$ unreduced

Let $X_+ = X \cup \{\text{disjoint basepoint}\}$

Set $h^n(X) = \tilde h^n(X_+)$ is an unreduced theory.

PF Recall cellular cohomology $h^n(pt) = G$.

$$\begin{tikzcd}
\to \prod_{(n-1)\text{-cells}} G \arrow[r, "\text{degrees of attaching maps}"] \arrow[d, equal] & \prod_{n\text{-cells}} G \arrow[r, "\text{degrees of attaching maps}"] \arrow[d, equal] & \prod_{(n+1)\text{-cells}} G \arrow[r] \arrow[d, equal] & \cdots \\
H^{n-1}(X^{n-1}, X^{n-2}; G) \arrow[r] & H^n(X^n, X^{n-1}; G) \arrow[r] & H^{n+1}(X^{n+1}, X^n; G) \arrow[r] & \cdots
\end{tikzcd}$$

<u>Goal</u> Show h^n has the same complex

<u>PF</u> Start w/ the following picture.

$$\begin{tikzcd}
 & h^{n-1}(X^{n-1}, X^{n-2}) \arrow[rr, "\partial_{n-1}"] \arrow[dr] & & h^n(X^n, X^{n-1}) \arrow[rr, "\partial_n"] \arrow[dr] & & h^{n+1}(X^{n+1}, X^n) \arrow[dr] & \\
h^{n-2}(X^{n-2}) \arrow[ur] & & h^{n-1}(X^{n-1}) \arrow[ur] & & h^n(X^n) \arrow[ur] & & {}
\end{tikzcd}$$

- Arrows of the same color come from a les of a pair and compose to zero

- $\partial^2 = 0$ since we pass through one of these compositions

- $h^k(X^n, X^{n-1}) = \tilde{h}^k(X^n/X^{n-1}) = \tilde{h}^k(\bigvee_{\alpha \in A} S^n_\alpha) = \prod_\alpha h^k(S^n_\alpha) = \begin{cases} \prod_\alpha G & \alpha = n \\ 0 & \text{otherwise} \end{cases}$

- From les for (X^n, X^{n-1}), $h^k(X^n) \simeq h^k(X^{n-1})$ if $k \neq n, n-1$

- In particular for $k > n$, $h^k(X^n) = h^k(X^0) = 0$.

Now let's look at this les in more detail.

$$
\begin{tikzcd}
{} \arrow[r] & h^{n-1}(X^{n-1}, X^{n-2}) \arrow[rr, "\partial_{n-1}"] \arrow[dr, two heads] & & h^{n}(X^{n}, X^{n-1}) \arrow[rr, "\partial_{n}"] \arrow[dr, two heads] & & h^{n}(X^{n+1}, X^{n}) \arrow[r] \arrow[dr, two heads] & {} \\
h^{n-2}(X^{n-2}) \arrow[ur, "\delta_{n-2}"] & & h^{n-1}(X^{n-1}) \arrow[ur, "\delta_{n-1}"] \arrow[dr] & & h^{n}(X^{n}) \arrow[ur, "\delta_{n}"] \arrow[dr] & & \\
& h^{n-1}(X^{n}) \arrow[ur, hook] \arrow[d, equal] & & 0 = h^{n-1}(X^{n-2}) & h^{n}(X^{n+1}) \arrow[u, hook] \arrow[d, equal] & h^{n}(X^{n-1}) \arrow[d, equal] & \\
& h^{n-1}(X) & & & h^{n}(X) & 0 & \\
0 = h^{n-1}(X^{n}, X^{n-1}) \arrow[ur] & & & 0 = h^{n}(X^{n+1}, X^{n}) \arrow[ur] & & &
\end{tikzcd}
$$

- $h^n(X) = \ker(\delta_n)$

- $\operatorname{Im}(\partial_n) = \operatorname{Im}(\delta_n)$

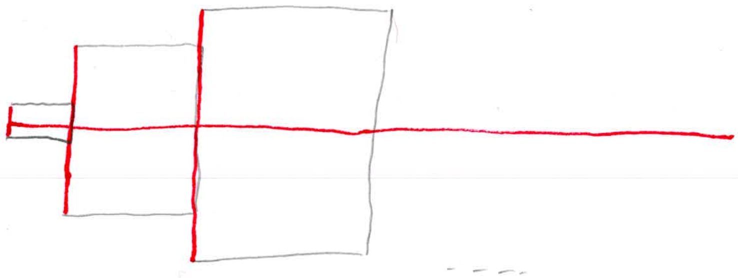
- $\ker(\partial_n) / \operatorname{Im}(\partial_{n-1}) = \ker(\partial_n) \Big/ \operatorname{Im}(\delta_{n-1}) = \ker(\delta_n) = h^n(X)$

<u>Note</u> Have not yet justified the assertion that $h^k(X^n) \simeq h^k(X)$ for $k<n$ yet.

We almost know the homology of the les is the expected thing. Still need to check: $h^k(X^n) = h^k(X)$ for $k < n$.

Mapping telescope $\bigcup_{i=k}^{\infty} X^i \times [i,\infty)/\sim \; = T$

$$X^1 \xrightarrow{j_1} X^2 \xrightarrow{j_2} X^3 \xrightarrow{j_3}$$



$T \underset{h.e.}{\simeq} X$

Let $Z = \left(\bigcup_i X^i \times \{i\}\right) \cup \left(\{x_0\} \times [k,\infty)\right)$ so $Z \underset{h.e.}{\simeq} \bigvee_{i=k}^{\infty} X^i$.

$T/Z = \bigvee_{i=k}^{\infty} \Sigma X^i$. (co)homology of Z and T/Z are shifted by a dimension.

$$\begin{array}{ccccccccc}
h^{k+1}(T,Z) & \longleftarrow & h^k(Z) & \xleftarrow{(\iota_i)^*} & h^k(T) & \longleftarrow & h^k(T,Z) & \longleftarrow & h^{k-1}(Z) \\
\cong & & \cong & & \cong & & \cong & & \cong \\
\prod h^{k+1}(\vee X_i) & \longleftarrow & \prod h^k(X^i) & \longleftarrow & h^k(X) & \xleftarrow{0} & \prod h^k(\vee(\Sigma X^i)) & \longleftarrow & \prod h^{k-1}(X_i)
\end{array}$$

$\uparrow$

Considering the les for the pair $(X \times I, X \times \{0\} \oplus X \times \{1\})$ shows this is surjective.

$(\iota_1^* x, \iota_2^* x, \iota_3^* x, \dots)$

$\cong$

$(j_1^* j_2^* \iota_3^* x, j_2^* \iota_3^* x, \iota_3^* x, \dots)$

Inverse limit stabilizes since the groups $H^k(X^n)$ do as $n \to \infty$.

Finally need to show ∂_n are degree maps.

$$\begin{tikzcd}
h^n(D^n_\alpha, \partial D^n_\alpha) & \underset{\cong}{\overset{G}{\tilde h^{n-1}(\partial D^n_\alpha)}} \arrow[l, "\sim", "\partial"'] & \underset{\cong}{\overset{G}{\tilde h^{n-1}(S^{n-1}_\beta)}} \arrow[l, "\text{Attaching map}"'] \arrow[d, "q_\beta^*"] \\
\Pi G \; h^n(X^n, X^{n-1}) \arrow[u, "\text{Characteristic map}"] & \tilde h^{n-1}(X^{n-1}) \arrow[l, "\partial_n"'] \arrow[u] & \tilde h^{n-1}(X^{n-1}, X^{n-2}) \arrow[l, "q*"'] \\
& \underset{\Pi G}{h^{n-1}(X^{n-1}, X^{n-2})} \arrow[ul, "\partial_n"] \arrow[u] \arrow[uur, bend right, "q_\beta^*"'] & h^{n-1}(X^{n-1}/X^{n-2}, X^{n-2}/X^{n-2}) \arrow[l, "\sim"'] \arrow[u, "S"']
\end{tikzcd}$$

We need to check our notion of "degree" $f^*: h^n(S^n) \to h^n(S^n)$ matches the ordinary one. This is clear for 0, 1. Any map $S^n \to S^n$ is a multiple of the identity, so its a multiple of id_G. Need to check additivity:

<u>Propn</u> For any cohomology theory, given $f, g : \Sigma X \to K$, have $(f+g)^* = f^* + g^*$.

$$\begin{tikzcd}
h(\Sigma X) & h(\Sigma X \vee \Sigma X) \arrow[l, "c^*"'] \arrow[d, shift right, "i_1^* \oplus i_2^*"'] & h(K) \arrow[l, "(f \vee g)^*"'] \arrow[dl, dashed, "{(f^*u, g^*u)}"] & u \\
& h(\Sigma(X)) \oplus h(\Sigma(X)) \arrow[u, shift right, "q_1^* \oplus q_2^*"'] \arrow[ul, dashed, "f_*u + g_*u"] & &
\end{tikzcd}$$

$\rightsquigarrow h_*(X)$ and $H_*(X, G)$ have the same cellular chain cpx

$\Rightarrow h_n(X) \approx H_n(X; G)$. One can check this is natural.

<u>What is this good for?</u>

① Universal relations on H^*

e.g. $H^1(X) = \langle X, S^1 \rangle$ For α a generator of $H^1(S^1; \mathbb{Z})$

$f^*(\alpha) \leftarrow\!\shortmid f$

$\alpha^2 = 0 \Rightarrow$ For any $\beta \in H^1(X)$, $\beta^2 = 0$.

<u>Compare</u> $H^1(X; \mathbb{Z}_2) = \langle X, \mathbb{R}P^\infty \rangle$ for α a generator of $H^1(\mathbb{R}P^\infty; \mathbb{Z}_2)$

$f^*(\alpha) \leftarrow\!\shortmid f$

② Cohomology Operations

Given $\beta \in H^m(K(G,n); G')$ and $H^n(X; G) \longleftrightarrow \langle X, K(G,n) \rangle$

$\alpha \longleftrightarrow f$

Pull back β to get $f^*\beta \in H^m(X; G')$. Get a natural map

$H^n(X; G) \longrightarrow H^m(X; G')$

$\alpha \longmapsto f^*(\beta)$.

(Very important to understanding $H^*(K(G,n))$).