Lecture 5

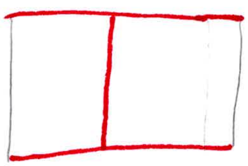
Last Time $\Omega X = \{ \gamma : [0,1] \to X : \gamma(0) = \gamma(1) = x_0 \} \subseteq MAP(I X)$

$$\langle \Sigma X, Y \rangle = \langle X, \Omega Y \rangle$$

$$\begin{tikzcd} \Omega X \arrow[r] & P_X \arrow[d] \\ & X \end{tikzcd}$$

Propn For any X and Y, $\langle \Sigma X, Y \rangle \simeq \langle X, \Omega Y \rangle$ is a group, and $\langle \Sigma^2 X, Y \rangle \simeq \langle X, \Omega^2 Y \rangle$ is an abelian group.

PP $\langle \Sigma X, Y \rangle$ are maps $X \times I \to Y$ taking red subset to basepoint.



Stack maps! If we have the double suspension, we look at maps $X \times I \times I \to Y$ and we can do the same htpy as we did for π_2 of a space.

Thm $\langle X, K(G,n) \rangle = H^n(X; G)$

Example $H^1(X; \mathbb{Z}) \simeq \langle X, K(\mathbb{Z}, 1) \rangle$

$\simeq \langle X, S^1 \rangle$

Specifically, $H^n(K(G,n); G) \simeq \mathrm{Hom}(H_n(K(G,n); \mathbb{Z}), G)$

$\simeq \mathrm{Hom}(G, G)$

continues

Let α be the identity map.

The map goes by

$$\langle X, \Omega(K(G,n))\rangle \longrightarrow H^n(X;G)$$
$$F \longmapsto F^*(\alpha)$$

<u>Worth noting</u> $\langle X, K(G,n)\rangle = [X, K(G,n)]$ for G abelian, $n>1$.

<u>Example of explicit proof</u>

<u>Case I</u> $G=\mathbb{Z}$, $n=1$, $K(G,n)\simeq S^1$. Claim $\langle X, S^1\rangle \simeq H^1(X;\mathbb{Z})$

Suppose X is simplicial and $[B]\in H^1(X;\mathbb{Z})$. Define $F: X \to S^1$ by

- On zero-simplices $f(\sigma^0)=$ basepoint
- On one-simplices $F|_{\sigma^1}$ winds $B(\sigma^1)$ times around S^1
- On two-simplices $F|_{\sigma^2}$, we know B is a cocycle, $B(\partial\sigma^2)=0 \Leftrightarrow f|_{\partial\sigma^2}$ is nulhomotopic $\Leftrightarrow F$ can be extended across σ^2
- Etc for $k>2$, $F|_{\partial\sigma^k}$ must be nulhomotopic

To complete proof: check this is independent of choices & that a coboundary goes to a nulhomotopic map.

This style works for general (G,n) & arbitrary CW cpxes, but...

Example $G = \mathbb{Z}$, $n = 2$ $K(\mathbb{Z}, 2) = \mathbb{C}P^\infty$

$$\begin{tikzcd} f^*(E) \arrow[r] \arrow[d] & E \arrow[d] \\ X \arrow[r, "f"] & \mathbb{C}P^\infty \end{tikzcd}$$

Maps to $\mathbb{C}P^\infty$, up to homotopy, are one dim'l vector bundles.

1st Chern class $\left\{ \begin{array}{c} \text{1-dimensional} \\ \text{bundles} \end{array} \right\} \Leftrightarrow \{ \text{second cohomology} \}$

Abstract Version

Defn A (reduced) cohomology theory $\tilde{h}_n(X)$ (for X a basepointed CW cpx) is

- A sequence of contravariant functors

$$\tilde{h}^n : \left\{ \begin{array}{c} \text{basepointed} \\ \text{CW cpxes} \end{array} \right\} \to \left\{ \begin{array}{c} \text{abelian} \\ \text{groups} \end{array} \right\}$$

- For X, A a CW pair, coboundary maps

$$\delta : \tilde{h}^n(A) \to \tilde{h}^{n+1}(X/A)$$

So that

① If $f \simeq g$ for $f, g : X \to Y$ then $f^* = g^* : \tilde{h}^n(Y) \to \tilde{h}^n(X)$ are the same.

② Long exact sequences for (X, A) a CW pair

③ Wedge sums: If $X = \bigvee_{\alpha} (X_\alpha)$, $\tilde{h}_n(X) = \prod_{\alpha} \tilde{h}_n(X_\alpha)$, with isomorphism given by $\prod (\iota_\alpha)_*$. [Automatic for finite wedge sums.]

④ [optional] $\tilde{h}^n(S^0) = 0$ for $n \neq 0$

For which collections of spaces $(K_n)_{n \in \mathbb{Z}}$ can $\langle X, K_n \rangle$ be a cohomology theory?

<u>Lemma</u> For any cohomology theory, $\tilde{h}^n(X) \simeq \tilde{h}^{n+1}(\Sigma X)$.

<u>PF</u> $\Sigma X = (X \times I) \big/ (x_0, t) \sim (x,1) \sim (x,0)$

$= C_0 X \big/ (x,1) \sim$ (basepoint

Consider $(C_0X, A) = (C_0X, X)$. Their les gives $\tilde{h}^n(X) = \tilde{h}^{n+1}(C_0X, X)$
$= h^{n+1}(\Sigma X)$

$$\begin{array}{ccc}
\underline{\text{Ergo}}\ \tilde{h}^n(\Sigma X) & \simeq & \tilde{h}^{n-1}(X) \\
\wr\| & & \wr\| \\
\langle \Sigma X, K_n \rangle & & \langle X, K_{n-1} \rangle \\
\wr\| & & \\
\langle X, \Omega K_n \rangle & &
\end{array}$$

Must have $K_{n-1} \overset{\text{whe}}{\simeq} \Omega K_n$

Defn A collection of CW-cpxes $(K_n)_{n\in\mathbb{Z}}$ is an Ω-spectrum if $K_{n-1} \simeq \Omega K_n$ weak htpy equivalence

- A collection $(K_n)_{n\in\mathbb{N}}$ is a spectrum if $\exists$ maps $\Sigma K_n \to K_{n+1}$ basepoint preserving.

- An Ω-spectrum is a spectrum: $(K_n \xrightarrow{whe} \Omega K_{n-1})$ $\leftrightarrow (\Sigma K_n \to K_{n+1})$.

Examples

- S^n is a (suspension) spectrum. (More generally, $\Sigma^n X$ for any X.)

- $K(G, n)$ is an Ω-spectrum.

Eg $\pi_k(K) = \lim \pi_{k+n}(K_n)$

$$
\left.
\begin{array}{rl}
\pi_{k+n}(K_n) = & \langle S^{k+n}, K_n \rangle \\
& \downarrow \\
& \langle S^{k+n+1}, \Sigma K_n \rangle \\
& \downarrow \\
\pi_{k+n+1}(K_{n+1}) & \langle S^{k+n-1}, K_{n+1} \rangle
\end{array}
\right\}
$$

Computing a direct limit gives $\pi_n^s(K_n)$, which are the homotopy groups of the spectrum.

<u>Thm</u> K an Ω-spectrum, $\tilde{h}_n : X \longmapsto \langle X, K_n \rangle$ is a reduced

↑ basepointed CW cpxes ↑ abelian groups

cohomology theory.

<u>Thm</u> (Brown Representability) Any cohomology theory arises in this way.

<u>Thm</u> Any cohomology theory satisfying the dimension axiom ($\tilde{h}^n(S^0) = 0$ for $n \neq 0$) is isomorphic to the standard cohomology in $h^0(pt)$ coefficients.

<u>Examples</u>

- K-theory (classifying space of a unitary group)
- Cobordism (Thom spectra)

(• Bordism is a reduced <u>homology</u> theory $\pi_n^s(X \wedge K_n)$.)

<u>Easy part of the first theorem</u>

- $X \longmapsto \langle X, K_n \rangle$ is a functor w/ image an abelian group. It is contravariant & preserves homotopy

Moreover for $f : X \to Y$, $f^* : \langle X, K_n \rangle \to \langle Y, K_n \rangle$ is clearly a group homomorphism.

<u>Wedge sum</u> If $X = \bigvee_{\alpha \in A}$ then $\langle X, K_n \rangle = \prod \langle X_\alpha, K_n \rangle$ by definition of the wedge product, as desired.

<u>Difficult part</u> LES (X, A) CW pair.

<u>Claim</u> $\exists$ a "les" of topological spaces

$$\begin{tikzcd}
A \arrow[r, hook] & X \arrow[r] & X/A \arrow[dll, bend left] \\
\Sigma A \arrow[r] & \Sigma X \arrow[r] & \Sigma(X/A) \arrow[dll, bend left] \\
{} & &
\end{tikzcd}$$

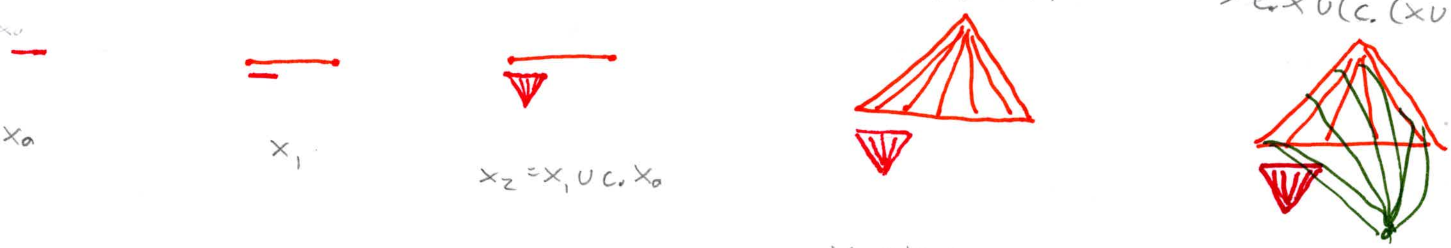
Puppe sequence

where any three terms are htpy equivalent to an inclusion/quotient triple.

<u>PF</u>

$$\begin{tikzcd}
A \arrow[r, hook] & X \arrow[r] & X/A \arrow[d, "\simeq"] \arrow[r, dashed, "?"] & \Sigma A \arrow[r, hook] & \Sigma X \arrow[r] & \Sigma(X/A) \\
& & X \cup CA \arrow[ur, "\text{collapse } X"'] & & &
\end{tikzcd}$$

$$\begin{tikzcd}
& & & \Sigma A \arrow[d, phantom, "\wr"] & \Sigma X \arrow[d, phantom, "\wr"] \\
A \arrow[r, hook] & X \arrow[r, hook] & X \cup C.A \arrow[r, hook] & C.X \cup C.A \arrow[r, hook] & C.X \cup (C.(X \cup C.A))
\end{tikzcd}$$



X_0 X_1 $X_2 = X_1 \cup C.X_0$ $X_3 = X_2 \cup C.X_1$

<u>Claim</u> From this sequence of spaces, get maps on homotopy classes $A \hookrightarrow X \longrightarrow X/A \longrightarrow \Sigma A$ which are exact as maps between $\langle\ , K\rangle$

$$\langle A, K\rangle \xleftarrow{e_1} \langle X, K\rangle \xleftarrow{e_2} \langle X/A, K\rangle \longleftarrow \langle \Sigma A, K\rangle$$

<u>PF</u> Suffices to show for first three terms.

<u>Claim</u> $\ker(e_1) = \operatorname{im}(e_2)$

$\ker(e_1) = \{\text{maps } F: X \to K \text{ st } F|_A : A \to K \text{ is nulhomotopic}\}$

$\operatorname{im}(e_2) = \langle X/A, K\rangle = \langle X \cup C.A, K\rangle$, or maps $X \to K$ that extend over $C.A$. This is the same thing.

<u>Thm</u> $\langle X, K_n\rangle$ is a cohomology theory for K_n an Ω-spectrum.

<u>PF</u> To show les on cohomology,

$$\tilde{h}(X/A) \longrightarrow \tilde{h}^n(X) \longrightarrow \tilde{h}^n(A) \longrightarrow \tilde{h}^{n+1}(X/A) \longrightarrow \tilde{h}^{n+1}(X)$$

$$\tilde{h}(X/A) = \langle \Sigma(X/A), K_{n+1}\rangle \to \langle \Sigma X, K_{n+1}\rangle$$

$$\tilde{h}^n(A) = \langle A, K_n\rangle = \langle A, \Omega K_{n+1}\rangle = \langle \Sigma A, K_{n+1}\rangle$$

$$\tilde{h}^{n+1}(X/A) = \langle X/A, K_{n+1}\rangle \to \langle X, K_{n+1}\rangle = \tilde{h}^{n+1}(X)$$

$$\langle \Sigma X, K_{n+1}\rangle \to \langle \Sigma A, K_{n+1}\rangle \to \langle X/A, K_{n+1}\rangle$$