Lecture 4

Propn For X cpt, Y a metric space, this is the topology of uniform convergence induced by $d(f,g) = \sup_{x \in X} d(f(x), g(x))$.

(Otherwise, "topology of uniform convergence on compact sets.")

Propn Y metric. $f_i \in MAP(X, Y)$ converges $\iff$ $f_i|_K$ converges uniformly on compact K.

Propn Y locally cpt.

ⓐ $ev : X \times Map(X, Y) \to Y$ is cts
$(x, f) \mapsto f(x)$

ⓑ $f : X \times Y \to Z$ is cts $\iff$ $\hat{f} : Y \to MAP(X, Z)$ is cts.
$y \mapsto f|_{X \times \{y\}}$

(Axioms of Cartesian closed categories)

Pf ⓐ Given $(x, f) \in X \times MAP(X, Y)$, $U \ni f(x)$ open on Y, wts $ev^{-1}(U)$ contains a nbhd of (x, f). f cts $\Rightarrow$ $f^{-1}(U)$ is an open nbhd of x. Also X is locally cpt, so there exists a cpt nbhd $K \ni x$ w/ $x \in Int(K)$.

Claim $Int(K) \times M(K, U) \subseteq ev^{-1}(U)$.

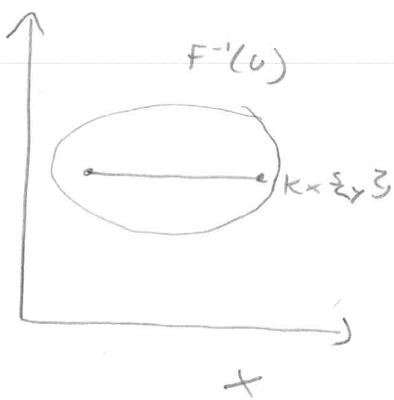
(b) $\Leftarrow$ Assume $\hat{F}$ is cts.

$$\begin{tikzcd} F : X \times Y \arrow[r, "id \times \hat{F}"] \arrow[rr, bend right, "cts"'] & X \times Map(X, Z) \arrow[r, "ev"] & Z \end{tikzcd}$$

$\Rightarrow$ Assume F is cts to see $\hat{F}$ is cts, need to show $\hat{F}^{-1}(M(K,U))$ is open. Let $y \in \hat{F}^{-1}(M(K,U))$.

<u>Know</u> $F(x,y) \in U$ for $x \in K$

$F^{-1}(U) \supseteq K \times \{y\}$



So $\exists$ $V \times W \subseteq F^{-1}(U)$ st $\begin{cases} V \subseteq X \text{ open} \\ W \subseteq Y \text{ open.} \end{cases}$ (Since K is cpt, can

take a finite subcover of $K \times \{y\}$ by product nbhds. Take unions for V and intersections for W.) Now $y \in W \subseteq \hat{F}^{-1}(M(K,U))$

<u>Propn</u> $MAP(X \times Y, Z) \longrightarrow MAP(Y, MAP(X,Z))$ is a homeomorphism if Y is Hausdorff and X is locally cpt and Hausdorff.

Now, recall any map $X \xrightarrow{F} Y$ is equivalent to an inclusion.

$$\begin{tikzcd} & Y \arrow[d, "\wr"] \\ X \arrow[ur, "F"] \arrow[r, hook] & M_F \end{tikzcd}$$

<u>Today</u> Any map is equivalent to a fibration.

<u>How?</u> $E_F = \{(x, \gamma) : F(x) = \gamma(0)\} \subseteq X \times MAP(I, Y)$

$x \in X$

$\gamma : [0,1] \to Y$

<u>What's the point?</u>

- Def retracts to X by contracting to a constant path
- Fibration over Y $\bar{F}: E_F$ (x, γ)

$$\begin{tikzcd} E_F \arrow[d] & (x,\gamma) \arrow[d, mapsto] \\ Y & \gamma(1) \end{tikzcd}$$

<u>Claim</u> This is a fibration and equivalent to $f : X \to Y$

<u>Pf</u> Let $i : X \hookrightarrow E_F$. Let $H : I \times E_F \longrightarrow E_F$

$x \mapsto (x, c_x)$

$(t, (x, \gamma)) \mapsto (x, \gamma_{[0, 1-t]})$.

Check H is cts? Projection onto X clearly is. Need to check $(I \times E_f \to Y^I)$. This is equivalent to

$(t, x, \gamma) \mapsto \gamma_{[0, 1-t]}$

$$(I \times E_f) \times I \xrightarrow{ev} Y$$ is cts, which is a

$$((t, x, \gamma), s) \mapsto \gamma_{[0,1-t]}(s) = \gamma(s(1-t))$$

restriction of $I \times (X \times Y^I) \times I \to Y$

$$(t, x, \gamma, s) \mapsto \gamma_{[0,1-t]}(s) = \gamma(s(1-t))$$ which is

cts because evaluation is cts. So H is a def retract of E_f onto X, and it takes $\bar{f}$ to f.

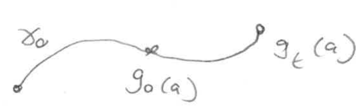
<u>Thm</u> (Milnor) If X, Y are CW cpxes, $f: X \to Y$, then E_f is htpy equivalent to a CW cpx.

Finally, to see $E_f \to Y$ is a fibration: Let

$$\begin{tikzcd} A \arrow[r, "\tilde{g}_0"] \arrow[d, hook] & E_f \arrow[d, "\bar{f}"] \\ A \times I \arrow[r, "g_t"'] & Y \end{tikzcd}$$

Let $\tilde{g}_0(a) = (h(a), \gamma_0(a))$. Set

$\tilde{g}_t = (h(a), \gamma_0(a) \cdot g_{[0,t]}(a))$.



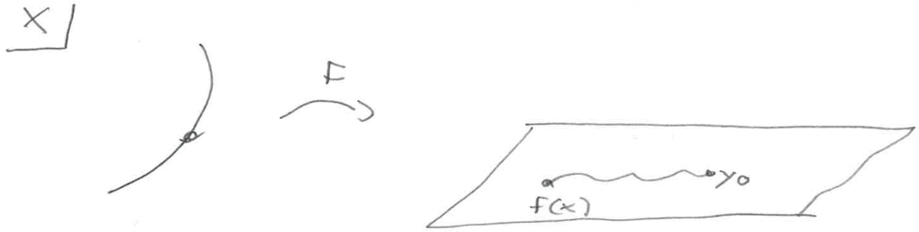
} Reparametrize appropriately

<u>Check this is cts</u>

$A \times I \to E_f \subseteq X \times Y^I$ is cts $\Leftrightarrow$ $A \times I \times I \to Y$ is cts

$(a, t) \mapsto \gamma_0(a) \cdot g_{[0,t]}(a)$ $\qquad$ $(a, t, s) \mapsto (\gamma_0(a) \cdot g_{[0,t]}(a))(s)$

<u>Defn</u> The homotopy fibre of $f: X \to Y$ is the fibre of $E_f \to Y$, i.e. $F_f = \bar{f}^{-1}(y_0) = \{(x, \gamma) : x \in X, \gamma(0) = f(x), \gamma(1) = y_0\}$.



Special Cases

① $i : A \hookrightarrow X$ an inclusion; $F_i = \{\gamma : \gamma(0) \in A, \ \gamma(1) = x_0\}$

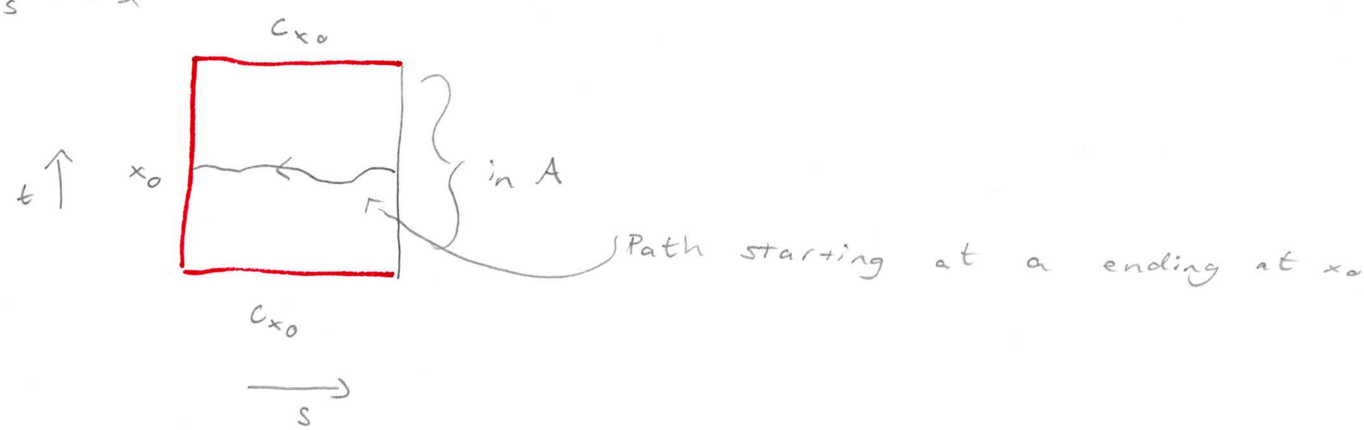
Propn $\pi_k(F_i) \cong \pi_{k+1}(X, A)$

Pf Same long exact sequence.

Why? Basepoint of $\pi_1(F_i)$ is a constant path at $x_0 \in A \subseteq X$

Element of $\pi_1(F_i)$ is a map $I_t \to Map(I_s, X) \iff$

$I_t \times I_s \to X$



What if A is a point? $\pi_{k+1}(X, A) = \pi_{k+1}(X) = \pi_k(F_i)$

Defn The based loop space of X, ΩX, is $\{\gamma : \gamma(0) = \gamma(1) = x_0\} \subseteq Map(I, X)$

We see $\pi_k(\Omega X) = \pi_{k+1}(X)$

Indeed
$$\begin{tikzcd} \Omega X \arrow[r] & MAP(I, X) = PX \arrow[d] \\ & X \end{tikzcd}$$

Examples
- $\Omega K(G,n) \simeq K(G,n-1)$
- $\Omega \mathbb{CP}^\infty \simeq S^1$
- $\Omega \mathbb{HP}^\infty \simeq S^3$

Iterating: Any $K(G,n)$ is an infinite loop space

$$\begin{tikzcd} K(G,n-1) \arrow[r] & PK(G,n) \arrow[d] \\ & K(G,n) \end{tikzcd}$$

Compare suspension $\pi_{k+1}(SX) = \pi_k(X)$ for $k < 2n+1$ when X is n-connected.

Suspension and loop space are adjoint functors.

Adjoint Functors

$\mathcal{C}, \mathcal{D}$ categories. Functors $F: \mathcal{C} \to \mathcal{D}$, $G: \mathcal{D} \to \mathcal{C}$ are adjoint if for $x \in \mathcal{C}$, $Y \in \mathcal{D}$, $Hom(Fx, Y)$ naturally isomorphic to $Hom(x, GY)$.

Examples

$$\begin{tikzcd} Set \arrow[r, bend left, "Free"] & Group \arrow[l, bend left, "Forget"] \end{tikzcd}$$

$Hom(FS, G) \simeq Hom(S, Forget\ G)$

$$\begin{tikzcd} \mathrm{Top} \arrow[r, bend left, "\times A"] & \mathrm{Top} \arrow[l, bend left, "\mathrm{Map}(A,\cdot)"] \end{tikzcd}$$

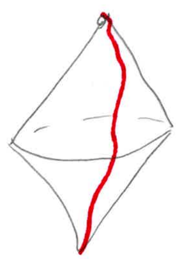
$\mathrm{Hom}(X \times A, Y) \simeq \mathrm{Hom}(X, \mathrm{MAP}(A,Y))$

for A locally cpt

$$\begin{tikzcd} \mathrm{Top}_\bullet \arrow[r, bend left, "\Sigma X"] & \mathrm{Top}_\bullet \arrow[l, bend left, "\Omega X"] \end{tikzcd}$$

$\mathrm{Hom}_\bullet(\Sigma X, Y) \simeq \mathrm{Hom}_\bullet(X, \Omega Y)$

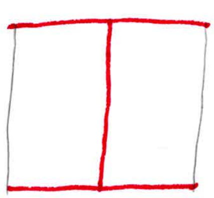
$X \times I/\sim \;\longrightarrow Y$



$\longrightarrow Y$ vs. $X \longrightarrow \Omega Y$

<u>Propn</u> For any X and Y, $\langle \Sigma X, Y\rangle \simeq \langle X, \Omega Y\rangle$ is a group, and $\langle \Sigma^2 X, Y\rangle \simeq \langle X, \Omega^2 Y\rangle$ is an abelian group.

<u>PF</u> $\langle \Sigma X, Y\rangle$ are maps $X \times I \longrightarrow Y$ taking red subset to basepoint.



I

$\leftarrow X \rightarrow$

Stack maps! If we have the double suspension, we look at maps $X \times I \times I \longrightarrow Y$ and we can do the same htpy as we did for π_2 of a space.