Propn. For $x$ cpt, $y$ a metric space, this is the topology of uniform convergence induced by $d(f,g) = \sup_{x \in X} d(f(x), g(x))$.

(Otherwise, "topology of uniform convergence on compact sets.")

Propn. $y$ metric, $f_i \in \text{Map}(x, y)$ converges $\iff f_i |_k$ converges uniformly on compact $k$.

Propn. $y$ locally cpt,

1. $ev : X \times \text{Map}(x, y) \to y$ is cts
   
   $(x, f) \mapsto f(x)$

2. $F : X \times y \to z$ is cts $\iff \hat{F} : y \to \text{Map}(x, z)$ is cts.
   
   $y \mapsto F |_{X \times \hat{y} z}$

(Axioms of Cartesian closed categories)

PF. (a) Given $(x, f) \in X \times \text{Map}(x, y)$, $U \supset F(x)$ open on $y$, wts $ev^{-1}(U)$ contains a nbhd of $(x, f)$. So cts $\iff F^{-1}(U)$ is an open nbhd of $x$. Also $x$ is locally cpt, so there exists a cpt nbhd $K \ni x$ w/ $x \in \text{Int}(K)$.

Claim. $\text{Int}(K) \times M(K, U) \subseteq ev^{-1}(U)$. 
(1) Assume \( \hat{F} \) is cts. \( F: X \times Y \xrightarrow{id_{\hat{F}}} X \times Map(Y, Z) \xrightarrow{ev} Z \) is cts.

(2) Assume \( F \) is cts \( \Rightarrow \) see \( \hat{F} \) is cts, need to show \( \hat{F}^{-1}(M(K, U)) \) is open. Let \( y \in \hat{F}^{-1}(M(K, U)) \).

\[
F^{-1}(U) = K \times \varepsilon_y B
\]

So \( V \times W \subseteq F^{-1}(U) \) s.t \( \bigcup V \subseteq X \) open. (Since \( K \) is open, can take a finite subcover of \( K \times \varepsilon_y B \) by product nbhds. Take unions for \( V \) and intersections for \( W \).) Now \( y \in W \subseteq \hat{F}^{-1}(M(K, U)) \).

\( \text{Prop} \quad Map(X \times Y, Z) \rightarrow Map(Y, Map(X, Z)) \) is a homeomorphism if \( Y \) is Hausdorff and \( X \) is locally cpt and Hausdorff.
Now, recall any map $X \xrightarrow{F} Y$ is equivalent to an inclusion,

$$\begin{array}{ccc}
X & \xrightarrow{F} & Y \\
\downarrow & & \downarrow \gamma \\
\mathbb{M} & \xleftarrow{} & \end{array}$$

Today Any map is equivalent to a fibration.

How? $E_f = \{ (x, t) : f(x) = y(t) \} \subseteq X \times \text{MAP}(I, Y)$

$x \in X$

$y : [0, 1] \rightarrow Y$

What's the point?

- Def $F$ retracts to $X$ by contracting to a constant path

- Fibration over $Y$ $\tilde{f} : E_f \rightarrow (x, y)$

Claim This is a fibration and equivalent to $f : X \rightarrow Y$

PF Let $c : x \leftrightarrow E_f$ . Let $H : I \times E_f \rightarrow E_f$

$x \mapsto (x, c_x)$

$(t, (x, y)) \mapsto (x, y \circ [0, 1-t])$

Check $H$ is cts? Projection onto $X$ clearly is. Need to check $(I \times E_f) \rightarrow Y^I$ . This is equivalent to

$(t, (x, y)) \mapsto y \circ [0, 1-t]$
\[(I \times E_F) \times I \rightarrow Y\] is cts, which is q

\[(t, x, s) \mapsto y_{[0, 1-t]}(s) = y(s(1-t))\]

restriction of \[I \times (x \times y^2) \times I \rightarrow Y\] which is cts because evaluation is cts. So \(H\) is a def retract of \(E_F\) onto \(X\), and it takes \(F\) to \(F\).

**Thm (Milnor)** If \(X, Y\) are CW cpxes, \(F: X \rightarrow Y\), then \(E_F\) is htpy equivalent to a CW cpx.

Finally, to see \(E_F \rightarrow Y\) is a fibration: Let

\[
\begin{array}{ccc}
A & \xrightarrow{\tilde{g}_0} & E_F \\
\downarrow & & \downarrow \tilde{F} \\
A \times I & \longrightarrow & Y \\
\end{array}
\]

Let \(\tilde{g}_0(a) = (h(a), \gamma_0(a))\). Set

\[
\tilde{g}_e = (h(a), \gamma_0(a), g_{[0, e]}(a)).
\]

Reparametrize appropriately

Check this is cts

\[A \times I \rightarrow E_F \subseteq X \times y^2\] is cts \(\Leftrightarrow\) \(A \times I \times I \rightarrow Y\) is cts

\[(a, t) \mapsto \gamma_0(a), \gamma_{[0, e]}(a)\]

DeF The homotopy fibre of \(F: X \rightarrow Y\) is the fibre of \(E_F \rightarrow Y\), i.e., \(F = \tilde{F}^{-1}(y_0) = \tilde{\gamma}(x, y): x \in X, \gamma(0) = F(x), \gamma(1) = y_0, \tilde{\gamma}^1\).
Special cases

1. \( i : A \hookrightarrow X \) an inclusion; \( F_i = \sum_{\gamma : \gamma(0) \in A} \gamma(1) = x_0 \)

**Prop.** \( \pi_k(F_i) \cong \pi_{k+1}(x_0, A) \)

**PF.** Same long exact sequence.

Why? Basepoint of \( \pi_1(F_i) \) is a constant path at \( x_0 \in A \subseteq X \).

Element of \( \pi_1(F_i) \) is a map \( I_0 \rightarrow \text{Map}(I_0, X) \).  

\[ I_0 = I \rightarrow X \]

Path starting at \( x_0 \).

What if \( A \) is a point? \( \pi_{k+1}(x_0, A) = \pi_k(F_i) \)

**Def.** The based loop space of \( X \), \( \Omega X \), is \( \sum_{\gamma : \gamma(0) = x_0} \gamma \in \text{Map}(I, X) \)

We see \( \pi_k(\Omega X) = \pi_{k+1}(x_0) \).
Indeed \[ \Omega X \longrightarrow \text{MAP}(\Sigma_j X) = \mathbb{P}X \]

\[
\downarrow
\]

\[
\downarrow
\]

Examples

\[
\Omega \mathbb{K}(G,n) \cong K(G,n-1)
\]

\[
\Omega \mathbb{O} \mathbb{P} \cong S^1
\]

\[
\Omega \mathbb{H} \mathbb{P} \cong S^3
\]

Iterating:

Any \( K(G,n) \) is

an infinite loop space

Compare suspension \( \pi_{k+1}(S X) = \pi_k(X) \) for \( k < 2n+1 \) when \( X \) is \( n \)-connected.

Suspension and loop space are adjoint functors.

Adjoint Functors

\( \mathcal{C}, \mathcal{D} \) categories. Functors \( F : \mathcal{C} \rightarrow \mathcal{D}, G : \mathcal{D} \rightarrow \mathcal{C} \) are adjoint

if for \( x \in \mathcal{C}, y \in \mathcal{D} \) \( \text{Hom}(Fx, y) \) naturally isomorphic to \( \text{Hom}(x, Gy) \).

Examples

\( \text{Free} \rightarrow \text{Group} \)

\( \text{Forget} \rightarrow \text{Set} \)

\( \text{Hom}(F_G, G) \cong \text{Hom}(S, \text{Forget } G) \)
\[
\text{Propn. For any } x \text{ and } y, \langle \Sigma x, y \rangle \cong \langle x, \Omega y \rangle \text{ is a group, and } \\
\langle \Sigma^2 x, y \rangle \cong \langle x, \Omega^2 y \rangle \text{ is an abelian group.}
\]

**Proof:** \langle \Sigma x, y \rangle \text{ are maps } x \times I \to y \text{ taking real subset to basepoint.}

Stack maps! If we have the double suspension, we look at maps \( x \times I \times I \to y \) and we can do the same htpy as we did for \( \pi_2 \) of a space.