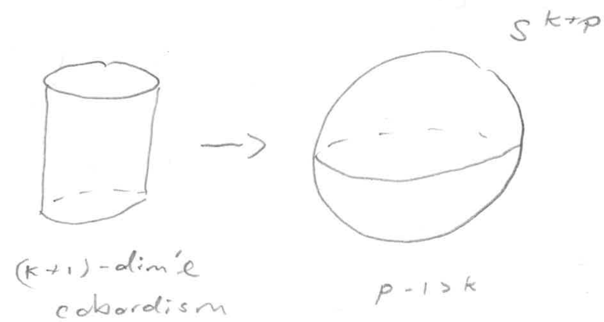
Lecture 3 Recall

$$\left\{\begin{array}{l}\text{(smooth) htpy} \\ \text{classes of} \\ \text{maps } f: M \to S^{\star}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{Framed } m\text{-}\star \text{ submfds} \\ \text{up to framed cobordism}\end{array}\right\}$$

Corollary

$$\pi_{k+p}(S^p) = \{\text{Framed cobordism classes of } k\text{'dim'l submanifolds in } S^{k+p}\}$$

For $k < p-1$, we have $\pi_k^S = \Omega_k^{Fr}$.



$p-1>k$

$k+p > 2k+1$

$k+p \geq 2k+2$

Left to prove

Thm 2 Two maps $M \to S^p$ are smoothly homotopic $\Leftrightarrow$ their Pontryagin mfds are framed cobordant.

Last time $\Rightarrow$

Today $\Leftarrow$

Finally, Thm 2 Say f, g both have y as a regular value.

Remains to show $\rightsquigarrow$ $(f^{-1}(y), f^*v) \sim (g^{-1}(y), g^*v) \Rightarrow f, g$ homotopic.

Lemma If $(f^{-1}(y), f^*v)$ is equal to $(g^{-1}(y), g^*v)$ then f is smoothly homotopic to g.

PF Let $N = f^{-1}(y)$. Since $f^*v = g^*v$, $df_x = dg_x \ \forall x \in N$.

Case I $f = g$ on a nbhd V of N. Let $h: S^p - y \to \mathbb{R}^p$ be projection.

Set
$$\begin{cases} H(x,t) = f(y) & \text{for } x \in V \\ H(x,t) = h^{-1}[t\,h(f(x)) + (1-t)\,h(g(x))] & \text{for } x \in M - N \end{cases}$$
These agree on $V - N$

Case II We need to arrange for such a neighborhood V first. Pick a product neighborhood $N \times \mathbb{R}^p \to V \subseteq M$, w/ V small enough so the antipode $\bar{y}$ of y is not in $f(V)$ or $g(V)$. Get $F, G : N \times \mathbb{R}^p \xrightarrow{V} \mathbb{R}^p \simeq S^p - \{\bar{y}\}$ w/ $F^{-1}(0) = G^{-1}(0) = N \times \{0\}$

and w/ $dF_{(x,0)} = dG_{(x,0)}$ = projection to $\mathbb{R}^p$. If we can find c st $F(x,u)\cdot u > 0$, $G(x,u)\cdot u > 0$ for all $x \in N$ and $0 < \|u\| < c$, then points in $F(x,u)$ and $G(x,u)$ are in the same open half-space in $\mathbb{R}^p$. So a htpy $(1-t)F(x,u) + t\,G(x,u)$ doesn't map any new points into 0 for $\|u\| < c$.

Let's find such a c.

Taylor's Thm $\|F(x,u) - u\| \le c_1 \|u\|^2$ for $\|u\| \le 1$ and some c_1

Approximation at $(x,0)$.

Cauchy-Schwarz $\Rightarrow |(F(x,u) - u)\cdot u| \le \|F(x,u) - u\|\,\|u\| \le c_1\|u\|^3$

$\|u\|^2 - F(x,u)\cdot u \le c_1\|u\|^3$

$\Rightarrow F(x,u)\cdot u \ge \|u\|^2 - c\|u\|^3 > 0$ on $0 < \|u\| < c = \mathrm{Min}(c_1^{-1}, 1)$

For distant points we can choose some $\lambda : \mathbb{R}^p \to \mathbb{R}$ with

$\lambda(u) = 1$ for $\|u\| \le \frac{c}{2}$

$\lambda(u) = 0$ for $\|u\| \ge c$

The homotopy $F_t(x,u) = [1 - \lambda(u)t]\,F(x,u) + \lambda(u)\,t\,G(x,u)$ is F for $\|u\| \ge c$, G for $\|u\| < \frac{c}{2}$, and has no new zeroes. F_1 can then ~~This can then~~ be deformed into G using case I.

Finally, Proof of Thm 2 ($\Leftarrow$) Say (x, w) is the framed cobordism between $f^{-1}(y)$ and $g^{-1}(y)$. Then just as we used the Pontrjagin-Thom construction to give a smooth map for any framed submfd last time, we can produce

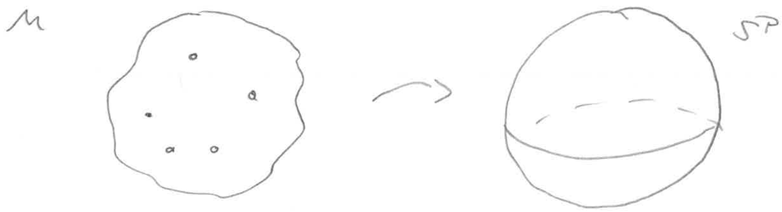
$F : M\times[0,1] \to S^p$

$(x,t) \mapsto F_t(x)$

w/ Pontrjagin mfd $(F^{-1}(y), F^*v) = (x, w)$. Now F_0, f have the same Pontrjagin mfd $\Rightarrow$ by the lemma, $F_0 \sim f$. Likewise $F_1 \sim g$. So $f \sim g$! $\square$

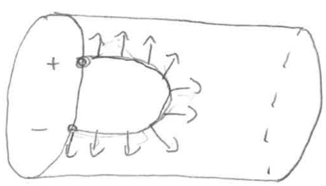
Example (Hopf degree)

M connected, orientable, $m = p$.



Framed submanifold Finite set of points w/ a basis for TM at each. Assign $\pm 1 = \text{sign}(x)$ at each according to whether the basis agrees w/ the orientation.

Framed cobordism Determined by sum of signs = degree.



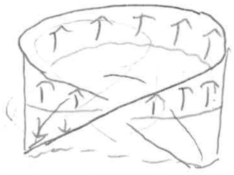
Thm (Hopf) Two maps $M^p \to S^p$ are smoothly homotopic $\Leftrightarrow$ they have the same degree.

So we see again that $\pi_0^s = \mathbb{Z}$.

HW What is π_1^s ?

What if M is not orientable?

· Given a basis for $T_x M$, can slide around a loop in M to reverse orientation.



Thm M connected, nonorientable, then two maps $M \to S^m$ are htpc $\Leftrightarrow$ they have the same mod 2 degree.

Last Thing The J-homomorphism

We have

$$\pi_r^S = \left\{ \begin{array}{c} \text{stably framed} \\ r\text{-mfds up to stable} \\ \text{framed cobordism} \end{array} \right\} = \Omega_r^{fr}$$

There is a map $J : \pi_r(SO) \longrightarrow \pi_r^S$

framings of S^r up to homotopy

all framed r-dim'l mfds up to framed cobordism

Concretely An element of $SO(q)$ is a map $S^{q-1} \to S^{q-1}$, so $\xi \in \pi_r(SO(q))$

In the sense of being a length-preserving map $\mathbb{R}^q \to \mathbb{R}^q$

gives $S^r \times S^{q-1} \to S^{q-1}$. This gives a map $S^{r+q} = S^r * S^q \to \Sigma(S^{q-1}) = S^q$.

$\Sigma(S^r \times S^{q-1}) = S^{r+q} \vee S^{r+1} \vee S^{q}$

The image of this map is an important subgroup of π_r^S.

	0	1	2	3	4	5	6	7	8	9	10	11
$\lvert\pi_r(SO)\rvert$	1	2	1	$\mathbb{Z}$	1	1	1	$\mathbb{Z}$	2	2	1	$\mathbb{Z}$
$\lvert Im(J)\rvert$	1	2	1	24	1	1	1	240	2	2	1	504
$\lvert\pi_r^S\rvert$	$\mathbb{Z}$	2	2	24	1	1	2	240	2^2	2^3	6	504

At 3 mod 4 order is $\frac{B_{2n}}{4n}$

Bernoulli number

For the past two lectures we've been talking about taking the suspension SX or ΣX of a space.

We've been talking about the maps

$$\langle X, Y\rangle \to \langle \Sigma X, \Sigma Y\rangle \to \langle \Sigma^2 X, \Sigma^2 Y\rangle \to \dots$$

We also want to think about a related construction, the based loop space, $\Omega X = \{\gamma : [0,1] \to X \; ; \; \gamma(0) = \gamma(1) = x_0\}$.

Before we talk seriously about this we should review topologies on spaces of maps.

Compact-open topology

- $MAP(X, Y)$ on Y^X
- Elements are cts maps. $f : X \to Y$
- Sub-basis for the topology : $M(K, U)$ for K compact in X and U open in Y.
- Basis is finite intersections of these

eg for $X = [0,1]$, a nbhd basis for $f : I \to Y$ is given by a partition of I into closed intervals $[x_i, x_{i+1}]$ and open sets U_i st $f([x_i, x_{i+1}]) \subseteq U_i$.