Lecture 29

Last Time w a complex vector bundle $\rightsquigarrow c_i(w) \in H^{2i}(B;\mathbb{Z})$

$$c(w) = 1 + c_1(w) + c_2(w) + \cdots + c_n(w)$$

$$c(w \oplus w') = c(w) \oplus c(w')$$

The Chern character

Want $K(X) \longrightarrow H^*(X;\mathbb{Z})$

Issue If we just make this map the total Chern class, then $\oplus \to \cup$.

Want a ring map $\oplus \to +$
$\otimes \to \cup$

To do this will have to move to $\mathbb{Q}$.

Splitting principle Suffices to use line bundles L.

$$\begin{array}{ccc} L_1 \oplus \cdots \oplus L_n & \longrightarrow & E \\ \downarrow & & \downarrow \\ Y & \longrightarrow & B \end{array} \qquad \begin{array}{c} L_1 \otimes L_2 \\ \downarrow \\ B \end{array} \qquad c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$$

$$H^*(Y) \hookleftarrow H^*(B)$$

What turns addition into multiplication? Exponentials

Defn $ch(L) = e^{c_1(L)} = 1 + c_1(L) + \dfrac{c_1(L)^2}{2!} + \dfrac{c_1(L)^3}{3!} + \cdots \in H^*(X;\mathbb{Q})$.

[Strictly speaking, the target is $\prod_{\wedge} H^\wedge(X;\mathbb{Q})$. This will not matter.]

Now say $E = L_1 \oplus \cdots \oplus L_n$. Let $c(L_i) = t_i$. We want
$$\downarrow$$
$$B$$

$$ch(E) = \sum_i ch(L_i)$$

$$= \dim(V) + (t_1 + \dots + t_n) + \frac{t_1^2 + \dots + t_n^2}{2} + \dots$$

It would be nice to express this in terms of $c(E)$ so we don't have to run through splitting every single time.

$$c(E) = (1+t_1) \cdots (1+t_n)$$

$$= 1 + \underbrace{(t_1 + \dots + t_n)}_{\sigma_1} + \underbrace{(t_1 t_2 + t_1 t_3 + \dots)}_{\sigma_2} + \dots + \underbrace{(t_1 \cdots t_n)}_{\sigma_n}$$

There exist Newton polynomials $s_1 = \sigma_1$ $\quad s_2 = \sigma_1^2 - 2\sigma_2$

$$s_k = \sigma_1 s_{k-1} - \sigma_2 s_{k-2} + \dots + (-1)^{k-2} \sigma_{k-1} s_1 + (-1)^{k-1} k \sigma_k$$

which have the property $s_k(\sigma_1, \dots, \sigma_n) = t_1^k + \dots + t_n^k$.

So $ch(E) = \displaystyle\sum_{k=0}^{\infty} \frac{s_k(c_1(E), \dots ; c_n(E))}{k!}$

Gives a ring homomorphism

$$ch : K^0(X) \longrightarrow H^{even}(X; \mathbb{Q})$$

$$\widehat{ch} : \tilde{K}^0(X) \longrightarrow \hat{H}^{even}(X; \mathbb{Q})$$

↑ we lose the degree 0 term.

Recall $K^1(X) = K^{-1}(X) = [X, U] = \langle X, U\rangle = \langle X, \Omega BU\rangle = \langle \Sigma X, U\rangle = \tilde{K}^0(\Sigma X)$

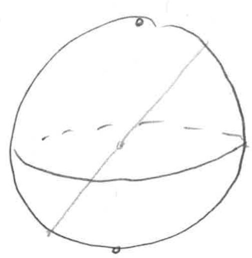
So we have

$$\begin{array}{ccc} \tilde{K}^0(\Sigma X) & \longrightarrow & \tilde{H}^{even}(\Sigma X; \mathbb{Q}) \\ \uparrow & & \wr\wr \\ K^1(X) & \longrightarrow & H^{odd}(X; \mathbb{Q}) \end{array}$$

Example $ch : \tilde{K}(S^{2n}) \hookrightarrow \tilde{H}^{2n}(S^{2n}; \mathbb{Q})$ w/ image $\tilde{H}^{2n}(S^{2n}; \mathbb{Z})$

$\subseteq \tilde{H}^{2n}(S^{2n}; \mathbb{Q})$.

Pf Consider S^2. $\tilde{K}^0(S^2) = \langle S^2, BU\rangle \simeq \pi_2(BU) = \pi_1(U) \simeq \mathbb{Z}$



Line bundles look like maps $[S^1, U(1)] \simeq \mathbb{Z}$

We already have a generator, the canonical line bundle H. (Because $c_1(H)$ generates $H^2(S^2; \mathbb{Z})$, and c_1 is additive.)

Consider $c_1(H \otimes H) = 2c_1(H)$ $\quad$ $c_1(H \oplus H) = 2c_1(H)$ $\}$ These things must be stably identified.

In fact we can work out the clutching ftn:

$\mathbb{C}P^\infty = \{[z_0 : z_1]\}$

$D^2_0 = \{[z : 1] : z := \frac{z_0}{z_1} \leq 1\}$ has a section $[z:1] \to (z, 1)$

$D^2_\infty = \{[1 : z^{-1}] : z = \frac{z_0}{z_1} \geq 1$ has a section $[1;z] \to (1, z^{-1})$ $\quad$ Clutching ftn is multiplication by z.

So H corresponds to $S^1 \hookrightarrow U(1)$,

$$z \mapsto [z]$$

Indeed, one can check directly that $z \mapsto \begin{pmatrix} z^2 & 0 \\ 0 & 1 \end{pmatrix}$ and

$z \mapsto \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}$ are homotopy equivalent, so $H \oplus H \sim (H \otimes H) \oplus 1$

$$2H = H^2 + 1$$

$\Rightarrow K^0(S^2) = \mathbb{Z}[H]/(H-1)^2$. Moreover $\tilde{K}^1(S^2) = 0$. So $K(S^2) = \mathbb{Z}[H]/(H-1)$

A rephrasing of Bott periodicity says for X compact there is an isomorphism

$$K^*(X) \otimes K^*(S^2) \xrightarrow{\sim} K^*(X \times S^2) \qquad [E]\otimes[E_2] \to [p_1^*(E_1)\, p_2^*(E_2)]$$

$$K^*(X) \otimes \mathbb{Z}[H]/(H-1)^2 \xrightarrow{\sim} K^*(X \times S^2) \qquad = [E_1] * [E_2]$$

$\exists$ a way of reducing to the smash product so that

$$\tilde{K}^*(X) \cong \tilde{K}^*(X) \otimes \tilde{K}^*(S^2) \longrightarrow \tilde{K}^*(X \wedge S^2)$$

K

$$\tilde{K}^*(A \wedge B) \to K^*(A \times B) \to K^*(A \vee B)$$
$$\wr\!\!\!\mid \quad K^*(A) \oplus K^*(B)$$
$$a * b \longrightarrow a * b \longrightarrow 0 \oplus 0$$

<u>So we have</u>

$$\begin{array}{ccc}
\tilde{K}(X) & \xrightarrow{\times (H-1)} & \tilde{K}(\Sigma^2(X)) \\
\downarrow \text{ch} & & \downarrow \text{ch} \\
\tilde{H}^*(X;\mathbb{Q}) & \xrightarrow{\times c_1} & \tilde{H}^{*+2}(\Sigma^2 X;\mathbb{Q})
\end{array}$$

] Iso via the isomorphism in $\mathbb{Z}$ coefficients.

So inductively the image of ch is always $\tilde{H}^*(X;\mathbb{Z})$

Corollary A class in $H^{2n}(S^{2n};\mathbb{Z})$ occurs as a Chern class $\Leftrightarrow$ it is divisible by $(n-1)!$

Pf $E \downarrow S^{2n}$ has $c(E) = 1 + c_n(E)$, so $ch(E) = \dim E + \frac{s_n(c_1,\ldots,c_n)}{n!}$

$= \dim E \pm \frac{n\, c_n(E)}{n!}$

$[s_n = \sigma_1 s_{n-1} + \cdots + (-1)^{n-1} n \sigma_n]$

2

This sort of thing doesn't happen in general; consider

$\gamma^1_{n,\mathbb{C}} = L \downarrow \mathbb{C}P^{n-1}$ $\quad ch(L) = 1 + h + \frac{h^2}{2} + \cdots + \frac{h^n}{n!}$ where h generates $H^2(\mathbb{C}P^n;\mathbb{Z})$.

Propn $K^*(X) \otimes \mathbb{Q} \xrightarrow{\sim} H^*(X;\mathbb{Q})$ whenever X is a finite CW cpx.

Pf Certainly true if X is a 0-cell. We induct on the total number of cells. Let $A = X$ - (a single cell). We have the Puppe sequence

$$X/A \longrightarrow \Sigma A \longrightarrow \Sigma X \longrightarrow \Sigma(X/A) \longrightarrow \Sigma^2 A \longrightarrow \cdots$$

(same number of cells as A: $K(\Sigma A)\otimes\mathbb{Q}$ and $K(\Sigma^2 A)\otimes\mathbb{Q}$)

$$\begin{array}{ccccccccc}
K(X/A) & \longleftarrow & K(\Sigma A)\otimes\mathbb{Q} & \longleftarrow & K(\Sigma X)\otimes\mathbb{Q} & \longleftarrow & K(\Sigma(X/A))\otimes\mathbb{Q} & \longleftarrow & K(\Sigma^2 A)\otimes\mathbb{Q} \\
\downarrow ch & & \downarrow ch & & \downarrow ch & & \downarrow ch & & \downarrow ch \\
H^*(X/A;\mathbb{Q}) & \longleftarrow & H^*(\Sigma A;\mathbb{Q}) & \longleftarrow & H^*(\Sigma X;\mathbb{Q}) & \longleftarrow & H^*(\Sigma(X/A);\mathbb{Q}) & \longleftarrow & H^*(\Sigma^2 A;\mathbb{Q}) \\
\text{sphere} & & & & & & \text{sphere} & &
\end{array}$$

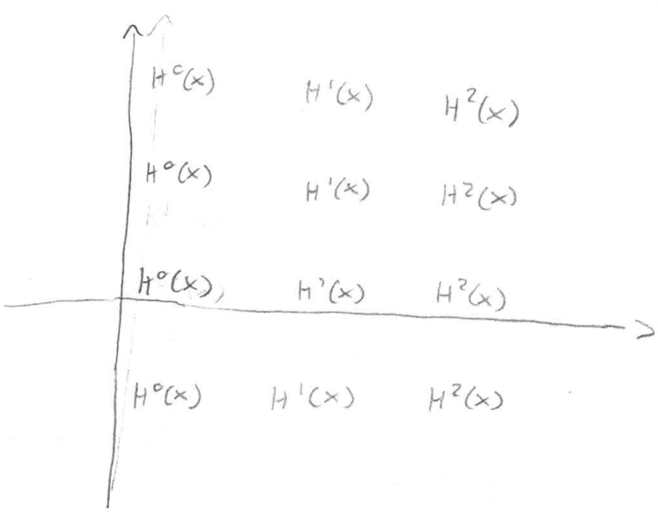
Five lemma: Desired iso for ΣX, hence for $\Sigma^2 X$, hence for X by Bott Periodicity.

Propn If X a finite CW cpx, $H^*(X;\mathbb{Z})$ is torsion free, then for ξ a bundle over X, $c(\xi)=1 \Rightarrow \xi$ is stably trivial.

PF We want to eliminate the possibility of torsion in $K(X)$. (This is the only way we could fail, since we have the iso over $\mathbb{Q}$.)

Recall Atiyah-Hirzebruch: For any extraordinary cohomology theory $h^i(X)$, there is a spectral sequence w/ $E_2^{p,q} = H^p(X; h^q(pt))$ converging to $h^*(X)$.

Now $K^q(pt) = \begin{cases} \mathbb{Z} & q \text{ even} \\ 0 & q \text{ odd} \end{cases}$



$E_2^{p,q} = H^p(X; K^q(pt))$

$= \begin{cases} H^p(X) & q \text{ even} \\ 0 & q \text{ odd} \end{cases}$

Note this is free.

$$\sum_{p+q=i} rk(E_\infty^{p,q}) = rk(K^i(X)) = \sum_{\substack{p=i \\ \text{mod } 2}} H^p(X) = \sum_{p+q=i} rk(E_2^{p,q})$$

How do you get torsion? From a cancelling arrow in the spectral sequence <u>but that kills rank</u> and therefore never happens.

<u>Note</u> Torsion summands, if they exist, can differ.

$\mathbb{RP}^7$: $\swarrow$ $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ $\searrow$ $\mathbb{Z}_8$

$\mathbb{RP}^3 \times \mathbb{RP}^5$: $\swarrow$ $(\mathbb{Z}_2)^{\oplus 5}$ $\searrow$ $\mathbb{Z}_4 \oplus \mathbb{Z}_2^{\oplus 2}$