# Lecture 28

## Complex Vector Bundles & the Chern Classes

We've previously mostly been working w/ $F = \mathbb{R}^n$. Today, $F = \mathbb{C}^n$,

$$\omega \begin{cases} F \hookrightarrow E \\ \quad\ \downarrow \\ \quad\ B \end{cases}$$ , and every point b has a neighborhood $U$ st

$\pi^{-1}(U) \cong U \times \mathbb{C}^n$ and overlap maps are linear.

Example One can get a complex bundle via equipping a real $2n$-plane bundle $\xi$ w/ a continuous $J : E(\xi) \to E(\xi)$ st $J^2 = -Id$, and $J$ preserves the fibres of $\xi$.

Then each fibre is a complex fector space via

$(x+iy)v = xv + J(yv)$.

Likewise any complex n-plane $\omega \rightsquigarrow \omega_{\mathbb{R}}$ the underlying $2n$-plane real vector bundle.

A complex structure $J$ on $TM$ is an <u>almost complex structure</u> on the manifold.

[It is an actual complex structure if every point on $M$ has a nbhd $U$ st $U \xrightarrow{h} \mathbb{C}^n$ such that $dh \circ J = J_0 \circ dh$. This is in general much harder to achieve.]

Examples ① $\mathbb{C}P^n$ is a complex mfd, and we have a canonical complex line bundle $\gamma^1_{n,\mathbb{C}}$ defined the way you expect.

② Recall complex line bundles are in general $\langle B, BU(1)\rangle = \langle B, \mathbb{C}P^\infty\rangle = H^2(B;\mathbb{Z})$. So for example we only get the trivial bundle over the circle. Suggests a relationship to orientations.

Lemma If $\omega$ is a complex vector bundle, then $\omega_{\mathbb{R}}$ has a canonical orientation.

Pf Let $V$ be a finite-dim'l cpx vector space. If $a_1, \ldots, a_n$ is a basis for $V$ over $\mathbb{C}$, then $a_1, ia_1, \ldots, a_n, ia_n$ is a basis for $V_{\mathbb{R}}$ determining the preferred orientation. This is not basis dependent because $GL(n,\mathbb{C})$ is connected $\Rightarrow$ we can pass from any such basis to another via cts deformation. Doing this to all of $\omega$ orients $\omega_{\mathbb{R}}$.

$\Rightarrow e(\omega_R) \in H^{2n}(B;\mathbb{Z})$ is well-defined. Furthermore note $e((\omega \oplus \omega')_{\mathbb{R}}) = e(\omega_{\mathbb{R}})\, e(\omega_{\mathbb{R}}')$ via

$$\begin{array}{ccc}
(a_1,\ldots,a_n),\ (b_1,\ldots,b_m) & \longrightarrow & (a_1, ia_1, \ldots, a_n, ia_n),\ (b_1, ib_1, \ldots, b_m, ib_m) \\
\downarrow & & \\
(a_1,\ldots,a_n, b_1,\ldots,b_m) & \longrightarrow & (a_1, ia_1, \ldots, a_n, ia_n, b_1, ib_1, \ldots, b_m, ib_m)
\end{array}$$

Instead of a Euclidean metric, for a complex bundle we want a Hermitian metric; i.e. one such that for $v \to |v|^2$, we have $|iv| = |v|$. The inner product

$$\langle v, w\rangle = \tfrac{1}{2}\left(|v+w|^2 - |v|^2 - |w|^2\right)$$
$$+ \tfrac{1}{2} i\left(|v+iw|^2 - |v|^2 - |iw|^2\right)$$

is linear in the first term & conjugate linear in the second. Moreover $\langle w, v\rangle = \overline{\langle v, w\rangle}$.

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$B$ paracompact $\rightsquigarrow$ every cpx bundle admits a Hermitian metric.

## Two constructions of the Chern classes

### ① Leray - Hirsch

$$w \left\{ \begin{tikzcd} F \arrow[r, hook] & E \arrow[d] \\ & B \end{tikzcd} \right.
\qquad
\begin{tikzcd} P(F) \arrow[r, hook, "j"] & P(E) \arrow[d] \\ & B \end{tikzcd}
\qquad
\begin{tikzcd} \gamma^1_{n,\mathbb{C}} \arrow[r] \arrow[d] & \gamma^1_{P(E)} \arrow[d] & \\ P(F) \arrow[r, hook, "j"'] & P(E) \arrow[r] & B \end{tikzcd}$$

$P(F) \cong \mathbb{C}P^{n-1}$

$\cong \mathbb{Z}[h]/h^n$

$\deg(h) = 2$

Consider $e(\gamma^1_{P(E)})$. This has $j^*(e(\gamma^1_{P(E)})) = e(\gamma^1_{1,\mathbb{C}})$. But

$$\begin{tikzcd} \gamma^1_{n-1,\mathbb{C}} \arrow[r] \arrow[d] & \gamma^1_{\mathbb{C}} \arrow[d] \\ \mathbb{C}P^{n-1} \arrow[r] & \mathbb{C}P^\infty \end{tikzcd}$$ is

just an inclusion and the class corresponding to the map (the Euler class) is a generator. Let $x = -e(\gamma^1_{P(E)})$, and let $h = j^*(x)$. This gives the same set up we had for the Stiefel-Whitney classes;

$$x^n = \underline{c^n}\,1 + \underline{c^{n-1}}\,x + \cdots + c_1 x^{n-1}$$

Clearly natural & has a product theorem.

## Second Construction

Use the splitting idea from last time.

$$\begin{array}{ccccc} (b,v,v) & \{b, v, w : v \neq 0\} = p_0^*(E) & \longrightarrow & E \\ \uparrow & & \downarrow & & \downarrow \\ (b,v) & \{b, v : v \neq 0\} \quad E_0 & \xrightarrow[p_0]{} & B \end{array}$$

Let $w_0$ be the orthogonal complement to this section

$$\begin{array}{cc} \{(b,v,w) : v, w \in F_b, v \neq 0, w \perp v\} = E(w_0) \\ \downarrow \\ E_0 \end{array}$$

Now look at the Gysin sequence

$$\cdots \longrightarrow \underset{\substack{\| \\ 0}}{H^{i-2n}(B;\mathbb{Z})} \xrightarrow{\cup e} H^i(B;\mathbb{Z}) \xrightarrow[\sim]{p_0^*} H^i(E_0;\mathbb{Z}) \longrightarrow \underset{\substack{\| \\ 0}}{H^{i-2n+1}(B)} \longrightarrow \cdots$$

$i < 2n$

**Defn** Let $\begin{array}{c} \xi \\ \downarrow \\ B \end{array}$ an $n$-plane cpx vector bundle. Then $c_n(V) = e(V)$ and

$c_i(V) = (p_0^*)^{-1}(c_i(w)) \in H^{2i}(B;\mathbb{Z})$

Once again we have a total chern class $c(\xi) = 1 + c_1(\xi) + \dots + c_n(\xi)$.

**Nontriviality** For the canonical line bundle on $\mathbb{C}P^{n-1}$, we have

$$\to H^{i+1}(V_0;\mathbb{Z}) \to H^i(\mathbb{C}P^{n-1};\mathbb{Z}) \xrightarrow{\cup c_1} H^{i+2}(\mathbb{C}P^{n-1};\mathbb{Z}) \xrightarrow{p_0^*} H^{i+2}(V_0;\mathbb{Z})$$

But $V_0 = (\text{line in } \mathbb{C}^n, \text{ vector on the line}) \simeq \mathbb{C}^n - \{0\}$

$$\simeq S^{2n-1}$$

Ergo for $0 \le i \le 2n-4$, we have $0 \to H^i(\mathbb{C}P^{n-1}) \xrightarrow{\cup c_1} H^{i+2}(\mathbb{C}P^{n-1}) \to 0$

Since in particular this works for $i=0$, we see $c_1$ generates the cohomology ring $\mathbb{Z}[c_1]/(c_1^{\,n})$.

<u>Propn</u> $H^*(Gr_n(\mathbb{C}^\infty);\mathbb{Z}) \simeq \mathbb{Z}[c_1(\gamma^n_{\mathbb{C}}), \ldots, c_n(\gamma^n_{\mathbb{C}})]$, where $\gamma^n_{\mathbb{C}}$ is the complex universal bundle.

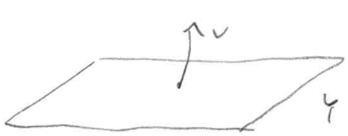
<u>Pf</u> First, we claim $H^*(E(\gamma^n_{\mathbb{C}})_0) \simeq H^*(Gr_{n-1}(\mathbb{C}^\infty))$

$$E(\gamma^n_{\mathbb{C}})_0 \longrightarrow Gr_{n-1}(\mathbb{C}^\infty)$$

$$\underset{v \neq 0}{(x, v)} \longrightarrow v^\perp \subseteq x \qquad \text{(std Hermitian metric)}$$

Look at the restriction $E'_N = E(\gamma^n_{\mathbb{C}})_0\big|_{G_n(\mathbb{C}^N)}$ to $Gr_n(\mathbb{C}^N)$, $N \gg 0$.

We have a map $F_N : E'_N \to \underset{\substack{\cup\!\!\!| \\ Y}}{Gr_{n-1}(\mathbb{C}^N)}$. Indeed $F_N^{-1}(Y) = (x, v)$ st $v \perp Y$




. This is an $(N-n+1)$-plane vector bundle w/ fibre $Y^\perp \subseteq \mathbb{C}^N$. It comes w/ a Gysin sequence

$$\cdots \to H^i(Gr_{n-1}(\mathbb{C}^N)) \to H^{i+2(N-n+1)}(Gr_{n-1}(\mathbb{C}^n)) \to H^{i+2(N-n+1)}(Gr_{n-1}(\mathbb{C}^n)) \to$$

Iso when $i \leq 2(N-n)$. Direct limit $\rightsquigarrow$ same result for $Gr_{n-1}(\mathbb{C}^\infty)$ and $E(\gamma^n_{\mathbb{C}})_0$.

So we now have a sequence

$$\cdots \to H^i(Gr_n(\mathbb{C}^\infty);\mathbb{Z}) \to H^{i+2n}(Gr_n(\mathbb{C}^\infty);\mathbb{Z}) \xrightarrow{(F_*)^{-1}p_0^*} H^{i+2n}(Gr_{n-1}(\mathbb{C}^\infty);\mathbb{Z}) \to H^{i+4n}(Gr_{n-1}(\mathbb{C}^\infty);\mathbb{Z}) \to \cdots$$

We wts $c_i(\gamma^n_{\mathbb{C}}) \mapsto c_i(\gamma^{n-1}_{\mathbb{C}})$. But we have

$$\begin{tikzcd}
\underset{(Y,w)}{\gamma^{n-1}_{\mathbb{C}} \oplus \varepsilon_{\mathbb{C}}} \arrow[d] & \underset{(x,v,w)}{p_0^*(\gamma^n_{\mathbb{C}})} \arrow[l] \arrow[r] \arrow[d] & \gamma^n_{\mathbb{C}} \arrow[d] \\
Gr_{n-1}(\mathbb{C}^\infty) & E(\gamma^n_{\mathbb{C}})_0 \arrow[l, "f"] \arrow[r, "p_0"'] & Gr_n(\mathbb{C}^\infty)
\end{tikzcd}$$

If we assume inductively that $H^*(Gr_{n-1}(\mathbb{C}^\infty);\mathbb{Z}) = \mathbb{Z}[c_1, \dots, c_{n-1}]$, we have

$$0 \to H^i(Gr_n(\mathbb{C}^\infty)) \xrightarrow{\cup c_n} H^{i+2n}(Gr_n(\mathbb{C}^\infty)) \to H^{i+2n}(Gr_{n-1}(\mathbb{C}^\infty)) \to 0$$

From which we obtain the desired result.

## Dual & Conjugate Bundles

$E \to B$ cpx, cpx structure $J$. Then $\overline{V}$ is the same underlying real vector bundle w/ complex structure $-J$.

The identity map $f: V \to \overline{V}$ is conjugate linear, i.e. $f(Je) = -Jf(e)$.

<u>Example</u> We claim $T\mathbb{C}P^1 \ncong \overline{T\mathbb{C}P^1}$ as complex vector bundles. For a map of the plane that reverses the complex structure of the plane is a reflection through

the sphere has no cts field of tangent lines.

**Lemma** $c_k(\bar{E}) = (-1)^k c_k(E)$.

**Proof** For a fibre $F$, pick a basis $v_1, \dots, v_n$. Then $v_1, iv_1, \dots, v_n, iv_n$ determines the preferred orientation for $F_{\mathbb{R}}$ and $v_1, -iv_1, \dots, v_n, -iv_n$ determines the " " for $\bar{F}_{\mathbb{R}}$. So $c_n(E) = e(E_{\mathbb{R}})$ $= (-1)^n e(\bar{E}_{\mathbb{R}}) = (-1)^n c_n(\bar{E})$. Lower degrees follow by induction.

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Also have a dual bundle $\mathrm{Hom}_{\mathbb{C}}(E, \mathbb{C})$. If $E$ has a Hermitian metric this is exactly $\bar{E}$ via

$$v \longmapsto \langle \, , v \rangle$$
$$\bar{E} \longrightarrow \mathrm{Hom}(E; \mathbb{C})$$

**Example** $T\mathbb{C}P^n$

**Claim** $c(T\mathbb{C}P^n) = (1+a)^{n+1}$ where $a$ generates $H^2(\mathbb{C}P^n; \mathbb{Z})$

(where $a = -c_1(\gamma^1_{n,\mathbb{C}})$).

**Pf** Let $\gamma^\perp = (\gamma^1_{n,\mathbb{C}})^\perp$ inside of $\mathbb{C}^{n+1}$. As with the real case,

$T\mathbb{C}P^n \cong \mathrm{Hom}_{\mathbb{C}}(\gamma^1_{n,\mathbb{C}}, \gamma^\perp)$. Now add a trivial bundle, obtaining

↑
maps from lines through the origin to the plane perpendicular

$$T\mathbb{C}P^n \oplus \underline{\mathbb{C}} = \mathrm{Hom}_{\mathbb{C}}(\gamma^1_{n,\mathbb{C}}, \gamma^\perp \oplus \gamma^1_{n,\mathbb{C}})$$
$$= \mathrm{Hom}_{\mathbb{C}}(\gamma^1_{n,\mathbb{C}}, \mathbb{C}^{n+1})$$
$$= \overline{(\gamma^1_{n,\mathbb{C}})}^{\oplus (n+1)}$$

Motivating question: How good are the Chern classes?

- Is there a bundle for each sequence $c_1, c_2, \ldots$? (geography)
- When do two bundles have the same sequence? (botany)
  - In particular, when does $c = 1 \Rightarrow$ stably trivial?

Example

$$\begin{array}{c} \gamma^1_7 \\ \downarrow \\ \mathbb{R}P^7 \end{array}$$

The complexification of a real vector bundle
$$\begin{array}{c} E \\ \downarrow \\ B \end{array}$$
is the bundle w/ fibre $F \otimes_{\mathbb{R}} \mathbb{C}$ $\forall$ fibre $F$.

$$\tilde{H}^* = \begin{cases} \mathbb{Z} & 7 \\ \mathbb{Z}_2 & 6 \\ 0 & 5 \\ \mathbb{Z}_2 & 4 \\ 0 & 3 \\ \mathbb{Z}_2 & 2 \\ 0 & 1 \\ 0 & 0 \end{cases}$$

Complexify $\omega : \gamma^1_7 \otimes_{\mathbb{R}} \mathbb{C}$
$$\begin{array}{c} \downarrow \\ \mathbb{R}P^7 \end{array}$$
, Look at $\omega^{\oplus 4}$

$$\begin{array}{ccc} 1+a & \overset{e}{\in} & H^2(\mathbb{R}P^7; \mathbb{Z}) \\ \downarrow & & \downarrow \\ 1+\bar{a} & & H^2(\mathbb{R}P^7; \mathbb{Z}_2) \end{array}$$

$\uparrow$ $w_2$

It's clear this should be nonzero.

$$(1+a)^4 = 1 + \underbrace{4a + 6a^2 + 4a^3}_{\text{even degree}} + a^4$$
$\uparrow$ degree 8

Thm (Karoubi)

$$\tilde{K}(\mathbb{R}P^7) = \mathbb{Z} \oplus \mathbb{Z}[w] \Big/ \begin{array}{l} (w-1)^2 = -2w \\ 8w = 0 \end{array}$$

$\uparrow$ $2^3$, where 3 is the # of positive even integers $< 7$.

$$\simeq \mathbb{Z} \oplus \mathbb{Z}_8$$

$\uparrow$ gen'd by $w-1$