Lecture 27

Wu's Formula for the Stiefel-Whitney Classes

Look at \( w_i(TM) \).

\[
S_i^i(u) = (p^* w_i) \cup u
\]

Moreover \( H^*(TM, \mathbb{Z}_2) \to H^*(M \times M, M \times M \setminus \Delta(M), \mathbb{Z}_2) \)

\[
u \mapsto u'
\]

Remember that \( w_i \mapsto w_i \).

So we have \( S_i^i(u') = (w_i \times 1) \cup u' \Rightarrow S_i^i(u'') = (w_i \times 1) \cup u'' \). Now the

clasp product map \( /B : H^*(X, \mathbb{Z}_2) \to H^*(X, \mathbb{Z}_2) \) is left \( H^*(x) \)-linear. So

we have

\[
((w_i \times 1) \cup u'')/\mu = w_i \cup (u''/\mu) = w_i = S_i^i(u'')/\mu.
\]

Prop. \( M \text{cpt, smooth} \Rightarrow \text{Stiefel-Whitney classes of } TM \text{ are given by}

the formula \( w_i = S_i^i(u'')/\mu \).

Corollary Two manifolds \( M \) the same homotopy type have the same

Stiefel-Whitney classes.

PF \( u'' \) is determined by a basis and dual basis. For \( H^*(M, \mathbb{Z}_2) \cong H^*(M, \mathbb{Z}_2) \)

Note that this is quite untrue of other characteristic classes.
Now consider
\[ H^{n-k}(M; \mathbb{Z}_2) \rightarrow \mathbb{Z}_2 \times 1 \rightarrow \langle Sq^k(x), u \rangle \]

\[\exists! v_k \in H^k(M; \mathbb{Z}_2) \text{ st } \langle v_k u, u \rangle = \langle Sq^k(x), u \rangle \text{ by Poincaré duality.}\]

The Wu class is \( v = v_1 + v_2 + \ldots + v_n \) satisfying \( \langle v u, u \rangle = \langle Sq^k(x), u \rangle \).

The total Stiefel-Whitney class \( w \) of \( TM \) is equal to \( Sq^k(v) \), i.e., \( w_k = \sum_{i+j=k} Sq^i(v_j) \).

PF: For any \( x \in H^r(M; \mathbb{Z}_2) \), \( x = \sum b_i \langle x, b_i \rangle \). Applying this to \( u \) we have
\[ v = \sum b_i \langle u u, b_i \rangle \]
\[ = \sum b_i \langle Sq^i(b_i), u \rangle \]
\[ \Rightarrow Sq^i(v) = \sum b_i \langle Sq^i(b_i), u \rangle \]
\[ = \sum (Sq^i(b_i) \times Sq^j(b_j))/u \]
\[ = Sq^w(u)/u \]
\[ = w \]
Obstructions & the Gysin Sequence

Last Time

Extending a $\kappa$-Frame $(s_1, \ldots, s_\kappa)$ nowhere dependent sections) over the $i$-skeleton is obstructed by maps

$$B = \bigotimes_{i=1}^{\infty} B^i \to V_k(F)$$

$s_i - 1 \to V_k(F)$

$\pi$ $\to V_k(F)$

$V_k(F)$ is $n-k-1$ connected, so the first potential obstruction lives in $\pi_{n-k}(V_k(F))$, and involves the extension over the $(n-k+1)$ skeleton.

Example: $n$-frames over $G^n$, 1-frames over $B^n$.

This gives a class $\eta_j (F) \in H^j (B; \bigotimes_{j=1}^{\infty} \wedge^{n-j+1} (F) \bigotimes)$, where $j = 1, \ldots, n-k+1$.

Formally:

Let $X$ be a locally path-connected topological space, and $M$ a module over some ring $R$. A local coefficients system of $R$-modules $\quad E$ w/ fiber $M$ is a bundle $\quad M \to E$ w/ action on $E$ by the

Fundamental groupoid of $X$. That is, for any path $\gamma$ in $X$ there is a morphism $\gamma: E_{s_{\gamma}} \to E_{t_{\gamma}}$, which only depends on the homotopy class of $\gamma$.

Cohomology is constructed by taking simplices in the usual way and using a coefficient in $E \otimes (\mathbb{R}_+)$. or by giving each cell a standard norm.
and using coefficients in that copy for that cell.

What are our coefficients?

If \( j > n - k + 1 \) is even and less than \( n \), we have \( \pi^*_{n-j+1}(V_{n-j+1}(\mathbb{F})) \cong \mathbb{Z}_2 \).

If \( j \) is odd or \( j = n \), \( \pi^*_{n-j}(V_{n-j}(\mathbb{F})) = \pi^*_{n-k}(V_k(\mathbb{F})) \) is \( \mathbb{Z} \), but noncanonically. There is however a canonical map to \( \mathbb{Z}_2 \).

**Prop.** The reduction mod 2 of \( o_j(\mathbb{F}) \) is \( w_j(\mathbb{F}) \).

**PF** \( H^*(Gr_{n}(\mathbb{R}^\infty), \mathbb{Z}_2) = \mathbb{Z}_2[w_1(x^n), \ldots, w_n(x^n)] \). Ergo \( h^* o_j(\mathbb{F}) \) is equal to some polynomial \( f_j(w_1(x^n), \ldots, w_n(x^n)). \) By naturality, this relationship is true for every single \( n \)-plane bundle. Furthermore since the polynomial has degree \( j \) in every term, we have

\[
f'_j(w_1, \ldots, w_n) = f'(w_1, \ldots, w_{j-1}) + A w_j
\]

where \( f' = f_{j,n} \) is a polynomial and \( j = j_{n} \) is 0 or 1.

But consider the \( n \)-plane bundle \( \mathbb{C}^{j_{n}+1} \otimes \mathbb{C}^{n-j_{n}} \) over \( Gr_{j_{n}}(\mathbb{R}^\infty) \).

Since there are \( n-j_{n}+1 \) nowhere dependent sections, \( o_{j}(\mathbb{F}) = 0 \), \( \Rightarrow h_{j_{n}} o_{j}(\mathbb{F}) = f'(w_1(x_{j_{n}}), \ldots, w_{j_{n}-1}(x_{j_{n}})) + A w_j(\mathbb{F}) = 0 \)

But the Stiefel–Whitney classes \( w_i(x_{j_{n}}) \) are linearly independent.

So \( f' \neq 0 \) and \( h \times o_j(\mathbb{F}) = -1 \) \( w_j(\mathbb{F}) \) in general. To show \( j_{n} = 1 \) it suffices
to show the existence of any vector bundle over a CW complex for which an \((n,j)
olomb{i}\) frame cannot be extended over the \(j\)-skeleton.

So, e.g., for \(j = n\) the perpendicular complement to the canonical line bundle over \(\mathbb{R}P^n\) will do. More generally the same bundle over \(\mathbb{R}P^n\) with \(e^{n+1}\) added on works for generally \(j\).

**The Gysin Sequence**

\[ F \to E \to E_0 \]

\[ p \quad \downarrow \quad p_0 \]

\[ B \]

**Thm.** To any oriented \(n\)-plane bundle \(E\), there is associated an exact sequence

\[ \cdots \to H^i(E) \xrightarrow{\partial} H^i(E_0) \xrightarrow{\partial^*} H^{i-1}(E) \to \cdots \]

in integer coefficients, where \(e = e(E)\) is the Euler class.

\[ F \to E \to E_0 \quad \quad \partial \]

\[ \cdots \to H^i(E,E_0) \to H^i(E) \to H^i(E_0) \xrightarrow{\partial} H^{i-1}(E,E_0) \to \cdots \]

*Use* \(u: H^{i-n}(E) \to H^i(E,E_0)\) to obtain

\[ \cdots \to H^{i-n}(E) \xrightarrow{g} H^i(E) \to H^i(E_0) \to H^{i-n+1}(E) \to \cdots \]

with \(g(x) = \langle x, u \rangle_E = x \cdot u(\text{cl}_E)\). Now \(H^i(E) \cong H^j(\emptyset)\), so

\[ \cdots \to H^{i-n}(\emptyset) \xrightarrow{\partial} H^j(\emptyset) \to H^j(E_0) \to \cdots \]

Since \(e = p^*(\text{cl}_E)\). \(\square\)
Similarly, for unoriented bundles and the Stiefel-Whitney class, an important example is:

Consider $V_n \to \text{H}^{i-1}(\mathbb{R}P^n) \overset{w_i}{\to} \text{H}^i(\mathbb{R}P^n) \to \text{H}^i(E_\circ) \to \ldots \to S^1 \to \text{H}^i(\mathbb{S}^n)

The space $E_\circ$ looks like $\mathbb{R}^{n+1}/\mathbb{S}^n$.

More generally:

2-fold covers $\to$ line bundles

**Corollary**: To any two-fold cover $\tilde{\mathcal{G}} \to \mathcal{G}$ there is associated an exact sequence.

$$\ldots \to \text{H}^{i-1}(\tilde{\mathcal{G}}) \overset{w_i}{\to} \text{H}^i(\mathcal{G}) \to \text{H}^i(\tilde{\mathcal{G}}) \to \text{H}^{i+1}(\mathcal{G}) \to \ldots$$

**Important example**: The oriented Grassmannian

$$\tilde{\text{Gr}}_n(\mathbb{R}^{\infty}) = \mathcal{O}_n \to \text{Gr}_n(\mathbb{R}^{\infty}) = \mathcal{O}_n$$

is a 2:1 cover.
The universal bundle lifts to an oriented n-plane bundle over \( G_n (\mathbb{R}^\infty) \):

\[
\begin{array}{c}
\tilde{G}_n (\mathbb{R}^\infty) \\
\downarrow \\
G_n (\mathbb{R}^\infty) \rightarrow B_n (\mathbb{R}^\infty)
\end{array}
\]

Propn \( \tilde{\text{H}}^* (\tilde{G}_n (\mathbb{R}^\infty); \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \text{w}_1 (\tilde{G}_n) \oplus \cdots \oplus \text{w}_n (\tilde{G}_n) \).

\( \Rightarrow \text{w}_1 (\tilde{G}_n) = 0 \), so \( \text{w}_1 (\tilde{G}_n) = 0 \) whenever \( \tilde{G}_n \) is orientable over a paracompact base. (Converse is h.w.)

\[
\begin{array}{c}
\text{PF} \\
\rightarrow \cdots \rightarrow \text{H}^{j-1} (\tilde{G}_n (\mathbb{R}^\infty)) \xrightarrow{\psi} \text{H}^j (\tilde{G}_n (\mathbb{R}^\infty)) \xrightarrow{p^*} \text{H}^j (G_n (\mathbb{R}^\infty)) \\
\text{line}
\end{array}
\]

where \( \psi \) is \( \text{w}_1 \), of the \( \tilde{G}_n \) bundle associated to the covering map. If \( c = 0 \), \( \text{H}^0 (G_n (\mathbb{R}^\infty)) \rightarrow \text{H}^0 (\tilde{G}_n (\mathbb{R}^\infty)) \rightarrow \text{H}^0 (G_n (\mathbb{R}^\infty)) \) implies \( G_n (\mathbb{R}^\infty) \) has two components. But any oriented \( n \)-plane in \( \mathbb{R}^\infty \) can be deformed easily to any other, so this is nonsense.

So \( c = \text{w}_1 (\tilde{G}_n) \), and none of the statement follows from the sequence.
Prop

The top obstruction class \( \alpha_0 (\xi) \in H^n (\theta, \mathcal{E} \xi, \nu, V (F) \xi) \) is the Euler class \( e(\xi) \).

Proof

Using orientation of the fibres, there is a canonical identification

\[ \mathcal{E} \xi \cong \mathcal{E} \xi (F) \cong H_{n-1} (F; \mathbb{Z}) \cong H_{n} (F, F_{\xi}; \mathbb{Z}) \]

isomorphic to \( \mathbb{Z} \). Now consider \( P_0 : E_0 \to B \) the restriction of \( p \) and the pullback bundle \( P_0^* (\xi) \) over \( E_0 \). This has a

\[ P_0^* (\xi) \to E \]

nowhere zero section via

\[ P_0^* E = S (b, v, u) : v \in E_b, v \to 0, u \in E_b \xi (b, v) \]

Using the Gysin sequence

\[ H^0 (\theta) \xrightarrow{\nu} H^0 (\theta) \xrightarrow{\mathcal{E} \xi} H^n (E_0) \]

\[ 1 \to \alpha_0 (\xi) \to 0 \]

So \( \alpha_0 (\xi) = 1 \nu e (\xi) \) for some \( 1 \). In particular, for \( \xi \) over \( \mathcal{E} \xi n \), we have

\[ \alpha_0 (\xi \mathcal{E} n) = \alpha_n (\xi n) \]

for some integer \( \alpha_n \), and the same equation is true for every single real oriented vector bundle. Moreover since \( \mod 2 \) this equation is

\[ \alpha_n (\xi \mathcal{E} n) = \frac{1}{2} \alpha_n (\xi n) \]

and \( \alpha_n (\xi \mathcal{E} n) \equiv 0 \), we have \( \alpha_n \) odd. If \( n \) is odd, \( e(\xi \mathcal{E} n) \) has order \( \frac{1}{2} \)
and we are done. If not, one checks the formula against the fact that $\chi(S^2) = 2$ (write down a vector field and check the map $S^{n-1} \to \mathbb{R}$ by hand).