Lecture 27

Wu's Formula for the Stiefel-Whitney classes

Look at $w_i(TM)$.

$$Sq^i(u) = (p^* w_i) \cup u$$

Moreover $H^*(TM, TM_0; \mathbb{Z}_2) \longrightarrow H^*(M\times M, M\times M - \Delta(M); \mathbb{Z}_2)$

$$u \longleftrightarrow u'$$

Remember that $w_i \times 1 = 1 \times w_i$.

So we have $Sq^i(u') = (w_i \times 1) \cup u' \Rightarrow Sq^i(u'') = (w_i \times 1) \cup u''$. Now the slant product map $/\mathcal{B}: H^*(X\times Y; \mathbb{Z}_2) \longrightarrow H^*(X; \mathbb{Z}_2)$ is left $H^*(X)$-linear. So we have

$$((w_i \times 1) \cup u'')/\mu = w_i \cup (u''/\mu) = w_i = Sq^i(u'')/\mu.$$

<u>Propn</u> M cpt, smooth $\Rightarrow$ Stiefel-Whitney classes of TM are given by the formula $w_i = Sq^i(u'')/\mu$.

<u>Corollary</u> Two manifolds w/ the same <u>homotopy type</u> have the same Stiefel-Whitney classes.

<u>Pf</u> u'' is determined by a basis and dual basis for $H^*(M_1; \mathbb{Z}_2) \cong H^*(M; \mathbb{Z}_2)$

Note that this is quite untrue of other characteristic classes.

Now consider

$$H^{n-k}(M;\mathbb{Z}_2) \longrightarrow \mathbb{Z}_2$$

$$x \longmapsto \langle Sq^k(x), \mu\rangle$$

$\exists!$ $v_k \in H^k(M;\mathbb{Z}_2)$ st $\langle v_k \cup x, \mu\rangle = \langle Sq^k(x), \mu\rangle$ by Poincaré duality.

The <u>Wu class</u> is $v = 1 + v_1 + \dots + v_n$ satisfying $\langle v \cup x, \mu\rangle = \langle Sq(x), \mu\rangle$.

<u>Thm</u> The total Stiefel-Whitney class w of TM is equal to $Sq(v)$,

i.e. $w_k = \sum_{i+j=k} Sq^i(v_j)$.

<u>Pf</u> For any $x \in H^{\pi}(M;\mathbb{Z}_2)$, $x = \sum b_i \langle x \cup b_i^{\#}, \mu\rangle$. Applying this to u we have

$$v = \sum b_i \langle v \cup b_i^{\#}, \mu\rangle$$

$$= \sum b_i \langle Sq(b_i^{\#}), \mu\rangle$$

$$\Rightarrow Sq(v) = \sum Sq(b_i) \langle Sq(b_i^{\#}), \mu\rangle$$

$$= \sum (Sq(b_i) \times Sq(b_i^{\#}))/\mu$$

$$= Sq(u'')/\mu$$

$$= w \quad \square$$

Obstructions & the Gysin sequence

Last Time

$$\begin{array}{c} F \longrightarrow E \\ \quad\downarrow \\ B = \{B^0 \subseteq B^1 \subseteq B^2 \subseteq \dots\} \end{array}$$

Extending a k-Frame ($s_1, \dots, s_k$ nowhere dependent sections) over the i-skeleton is obstructed by maps

$$(D^i, S^{i-1}) \longrightarrow V_k(\xi)\big|_{D^i} \simeq V_k(F)$$

$V_k(F)$ is $n-k-1$ connected, so the first potential obstruction lives in $\pi_{n-k}(V_k(F))$, and involves the extension over the $(n-k+1)$ skeleton.

- eg n-frames over B^1; 1-Frames over B^n.

This gives a class $o_j(\xi) \in H^j(B; \{\pi_{j-1} V_{n-j+1}(F)\})$, where $j = n-k+1$.

Formally:

Let X be a locally path-connected topological space, and M a module over some ring R. A local coefficients system of R-modules E w/ Fiber M is a bundle $M \hookrightarrow E \to X$ w/ action on E by the Fundamental groupoid of X. That is, for any path γ in X there is a morphism $\gamma_* : E_{\gamma_0} \to E_{\gamma_1}$ which only depends on the homotopy class of γ.

Cohomology is constructed by taking simplices in the usual way and using a coefficient in $E|_{\sigma(bp)}$ or by giving each cell a standard map and using coefficients in that copy for that cell.

What are our coefficients?

If $j = n-k+1$ is even and less than n, we have $\pi_{j-1}(V_{n-j+1}(F)) \simeq \mathbb{Z}_2$.

$$\pi_{j-1}(V_{n-j+1}(F)) = \pi_{n-k}(V_k(F))$$

If j is odd or $j=n$, $\pi_{j-1}(V_{n-j+1}(F)) = \pi_{n-k}(V_k(F))$ is $\mathbb{Z}$, but noncanonically. There is however a canonical map h to $\mathbb{Z}_2$.

<u>Propn</u> The reduction mod 2 of $o_j(\xi)$ is $w_j(\xi)$.

<u>Pf</u> $H^*(Gr_n(\mathbb{R}^\infty); \mathbb{Z}_2) = \mathbb{Z}_2[w_1(\gamma^n), \ldots, w_n(\gamma^n)]$. Ergo $h_* o_j(\gamma^n)$ is equal to some polynomial $f_j(w_1(\gamma^n), \ldots, w_n(\gamma^n))$. By naturality, this relationship is true for every single n-plane bundle. Furthermore since the polynomial has degree j in every term, we have

$$f_j(w_1, \ldots, w_n) = F'(w_1, \ldots, w_{j-1}) + d w_j$$

where $F' = F'_{j,n}$ is a polynomial and $d = d_{j,n}$ is 0 or 1.

But consider the n-plane bundle $\eta = \gamma^{j-1} \oplus \varepsilon^{n-j+1}$ over $Gr_{j-1}(\mathbb{R}^\infty)$.

Since there are $n-j+1$ nowhere dependent sections, $o_j(\eta) = 0$, $\Rightarrow$

$$h_j o_j(\eta) = F'(w_1(\gamma_{j-1}), \ldots, w_{j-1}(\gamma_{j-1})) + \underbrace{\cancel{d w_j(\eta)}}_{0} = 0$$

But the Stiefel-Whitney classes $w_i(\gamma_{j-1})$ are linearly independent.

So $F' \equiv 0$ and $h_* o_j(\xi) = d\, w_j(\xi)$ in general. To show $d=1$, it suffices to show the existence of <u>any</u> vector bundle over a CW cpx

for which an $(n-j+1)$-frame cannot be extended over the j-skeleton.

So, e.g., for $j=n$ the perpendicular complement to the canonical line bundle over $\mathbb{RP}^n$ will do. More generally the same bundle over $\mathbb{RP}^j$ w/ ε^{n-j} added on works for general j.

The Gysin Sequence

$$\begin{tikzcd} F \arrow[r, hook] & E \arrow[d, "p"'] & \supseteq E_0 \arrow[dl, "p_0"] \\ & B & \end{tikzcd}$$

<u>Thm</u> To any oriented n-plane bundle ξ, there is associated an exact sequence

$$\cdots \to H^i(B) \xrightarrow{\cup e} H^{i+n}(B) \xrightarrow{p_0^*} H^{i+n}(E_0) \to H^{i+1}(B) \xrightarrow{\cup e} \cdots$$

in integer coefficients, where $e = e(\xi)$ is the Euler class.

<u>Pf</u> $$\cdots \to H^j(E, E_0) \to H^j(E) \to H^j(E_0) \xrightarrow{\delta} H^{j+1}(E, E_0) \to \cdots$$

Use $\cup u : H^{j-n}(E) \to H^j(E, E_0)$ to obtain

$$\cdots \to H^{j-n}(E) \xrightarrow{g} H^j(E) \to H^j(E_0) \to H^{j-n+1}(E) \to \cdots$$

with $g(x) = (x \cup u)|_E = x \cup (u|_E)$. Now $H^j(E) \cong H^j(B)$, so

$$\cdots \to H^{j-n}(B) \xrightarrow{\cup e} H^j(B) \to H^j(E_0) \to \cdots$$

since $e = p^*(u|_E)$. $\square$

Similarly for unoriented bundles and the top Stiefel-Whitney class.

Important example

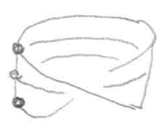
Consider $\begin{array}{c}\gamma^1_n\\ \downarrow \\ \mathbb{RP}^n\end{array}$ $\quad \cdots \to H^{j-1}(\mathbb{RP}^n) \xrightarrow{\cup w_1} H^j(\mathbb{RP}^n) \to H^j(E_0) \to \cdots$

$$H^j(E_0) \cong H^j(S^n)$$

The space E_0 looks like $\mathbb{R}^{n+1} - \{0\}$.

More generally

2-fold covers $\longleftrightarrow$ Line bundles



Corollary To any two-fold cover $\tilde{B} \to B$ there is associated an exact sequence

$$\cdots \to H^{j-1}(B) \xrightarrow{\cup w_1} H^j(B) \to H^j(\tilde{B}) \to H^j(B) \to \cdots$$

Important example

The oriented Grassmanian

$$\begin{array}{c}\tilde{Gr}_n(\mathbb{R}^\infty) = BSO(n) \quad \} \text{ Oriented } n\text{-planes}\\ \downarrow \\ G_n(\mathbb{R}^\infty) = BO(n)\end{array}$$

is a 2:1 cover.

The universal bundle lifts to an oriented n-plane bundle over $\widetilde{Gr}_n(\mathbb{R}^\infty)$.

$$\begin{tikzcd} \tilde{\gamma}^n \arrow[r] \arrow[d] & \gamma^n \arrow[d] \\ \widetilde{Gr}_n(\mathbb{R}^\infty) \arrow[r] & G_n(\mathbb{R}^\infty) \end{tikzcd}$$

Propn $\hat{H}^*(\widetilde{Gr}_n(\mathbb{R}^\infty); \mathbb{Z}_2) = \mathbb{Z}_2[w_2(\hat{\gamma}_n), \ldots, w_n(\tilde{\gamma}^n)]$.

$\Rightarrow w_1(\tilde{\gamma}^n) = 0$, so $w(\xi) = 0$ whenever ξ is orientable over a paracompact base. (Converse is hw)

PF $\cdots \to H^{j-1}(Gr_n(\mathbb{R}^\infty)) \xrightarrow{\cup c} H^j(Gr_n(\mathbb{R}^\infty)) \xrightarrow{p^*} H^j(\widetilde{Gr}_n(\mathbb{R}^\infty)) \to \cdots$

where c is w_1 of the line bundle associated to the covering map. If $c = 0$, $0 \to H^0(Gr_n(\mathbb{R}^\infty)) \to H^0(\widetilde{Gr}_n(\mathbb{R}^\infty)) \to H^0(Gr_n(\mathbb{R}^\infty)) \xrightarrow{\cup c}$

implies $\widetilde{Gr}_n(\mathbb{R}^\infty)$ has two components. But any oriented n-plane in $\mathbb{R}^\infty$ can be deformed ctsly to any other, so this is nonsense.

So $c = w_1(\gamma^n)$, and now the statement follows from the sequence.

Propn The top obstruction class $o_n(\xi) \in H^n(B, \{\pi_{n-1} V_1(F)\})$ is the Euler class $e(\xi)$.

PF Using orientation of the fibres, there is a canonical identification $\pi_{n-1} V_1(F) \simeq \pi_{n-1}(F-0) \underset{\uparrow \text{Hurewicz}}{\simeq} H_{n-1}(F-0; \mathbb{Z}) \simeq H_n(F, F_0 0; \mathbb{Z})$ becomes canonically isomorphic to $\mathbb{Z}$. Now consider $p_0 : E_0 \longrightarrow B$ the restriction of p and the pullback bundle $p_0^*(\xi)$ over E_0. This has a nowhere zero section via

$$\begin{array}{ccc} p_0^*(\xi) & \longrightarrow & E \\ \downarrow & & \downarrow \\ E_0 & \xrightarrow{p_0} & B \end{array}$$

$$p_0^* E = \{(b,v,u) : v \in E_b,\ v \neq 0,\ u \in E_b\} \qquad (b,v,v)$$
$$\downarrow$$
$$\{(b,v) : v \in E_b,\ v \neq 0\} \qquad (b,v)$$

Using the Gysin sequence

$$H^0(B) \xrightarrow{\cup e} H^n(B) \xrightarrow{\pi_0^*} H^n(E_0)$$
$$\lambda \longmapsto o_n(\xi) \longmapsto 0$$

So $o_n(\xi) = \lambda \cup e(\xi)$ for some λ. In particular for $\tilde{\gamma}_n$ over $\tilde{G}_n$, we have $o_n(\tilde{\gamma}^n) = \lambda_n e(\tilde{\gamma}_n)$ for some integer λ_n, and the same equation is true for every single real oriented vector bundle. Moreover since mod 2 this equation is $w_n(\tilde{\gamma}^n) = \underbrace{\lambda_n}_{\text{mod } 2} w_n(\tilde{\gamma}^n)$, and $w_n(\tilde{\gamma}^n) \neq 0$, we have λ_n odd. If n is odd, $e(\tilde{\gamma}^n)$ has order →

and we are done. If not, one checks the formula against the fact that $\chi(S^2) = 2$ (write down a vector field and check the map $S^{n-1} \to \mathbb{Z}$ by hand).